

Part 1: Depreciation

A new car is rarely resold for its original purchase price. As soon as a new car is driven away from a dealership, the value of the car decreases. This decrease in value is called **depreciation**. The show room price of a new car is \$26 000. The salesperson estimates that the value of the car will depreciate by an average of 10% each year.

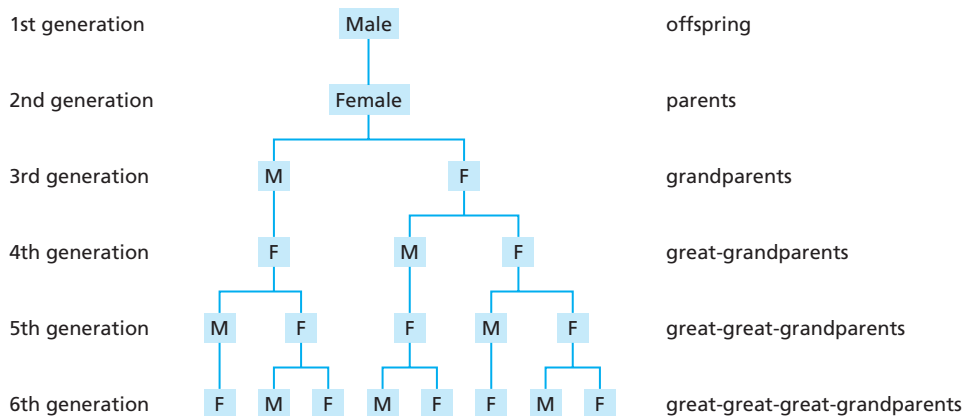


Think, Do, Discuss

1. Determine the dollar amount of depreciation for the first year.
2. Determine the value of the car at the end of the first year.
3. What percent of the original purchase price does the value in step 2 represent?
4. Determine the value of the car at the end of the second year.
5. Write the first five terms of the sequence that represents the value of the car at the end of each year for the first five years.
6. Let t_n represent any term of this sequence. How could you represent the term before t_n ?
7. For this sequence, write a formula that uses the previous term to calculate any term, except the first term.
8. At the end of which year will the car's value be about one-half of its original purchase price?

Part 2: The Fibonacci Sequence

There are many examples of sequences in nature, and in our everyday lives. A female honeybee hatches from an egg laid by a female honeybee, after the egg has been fertilized by the male. So each female honeybee has two parents. However, a male honeybee hatches from an unfertilized egg and has only one parent, a female honeybee. The tree diagram shows six generations of a typical male honeybee.



Think, Do, Discuss

1. Starting with the first generation, write the sequence of the number of honeybees in each generation.
2. Identify the pattern in this sequence.
3. Use this pattern to determine the next five terms of the sequence.
4. Let t_n represent any term in this sequence. How could you represent the term before t_n ? How could you represent the term before the term that is before t_n ?
5. Given that $t_1 = 1$, $t_2 = 1$, and n is a natural number, determine a formula for the general term of this sequence.
6. This sequence is known as the **Fibonacci sequence**, and it occurs in many different situations in nature. Could you easily calculate the 50th term of this sequence? Explain.
7. Create a tree diagram that represents six generations of a female honeybee. Write the sequence that represents the number of honeybees in each generation.
8. Does this sequence have the same pattern as the sequence in step 1? Is this sequence a Fibonacci sequence?
9. Determine a formula for this sequence.
10. Use your formula to find the next five terms of this sequence.
11. For 11 generations of honeybees, who has more ancestors, a male honeybee or a female honeybee?

Key Ideas

- In a **recursive sequence**, a new term is generated from the previous term or terms. For example, 1, 2, -1, 3, -4, 7, -11, ... is a recursive sequence because each term, beginning with the third term, is the result of subtracting t_{n-1} from t_{n-2} .
- A **recursive formula** shows how to find each term from the previous term or terms. For example,

$$t_1 = 1, t_2 = 2, t_n = t_{n-2} - t_{n-1}, \text{ where } n \in N$$

To find t_n , subtract t_{n-1} from t_{n-2} . The first two terms are given.

The sequence is 1, 2, -1, 3, -4, 7, -11, ...

- A recursive formula refers to at least one known term. The first term, and sometimes several other terms, appear with the formula. Examine the formula carefully before applying it.

Example 1

Write the first four terms of each sequence.

(a) $t_1 = 2, t_n = 3t_{n-1} + 5$ (b) $t_1 = -1, t_2 = 1, t_n = 2t_{n-2} + 4t_{n-1}$

Solution

- (a) The first term is given. To find the next term, multiply the previous term by 3 and add 5.

$$\begin{array}{llll}
 t_1 = 2 & t_2 = 3t_{2-1} + 5 & t_3 = 3t_{3-1} + 5 & t_4 = 3t_{4-1} + 5 \\
 & = 3t_1 + 5 & = 3t_2 + 5 & = 3t_3 + 5 \\
 & = 3(2) + 5 & = 3(11) + 5 & = 3(38) + 5 \\
 & = 6 + 5 & = 33 + 5 & = 114 + 5 \\
 & = 11 & = 38 & = 119
 \end{array}$$

The first four terms are 2, 11, 38, and 119.

- (b) The first two terms are given. To find the third term, multiply the first term by 2, multiply the second term by 4, and add.

$$\begin{array}{llll}
 t_1 = -1 & t_2 = 1 & t_3 = 2t_{3-2} + 4t_{3-1} & t_4 = 2t_{4-2} + 4t_{4-1} \\
 & & = 2t_1 + 4t_2 & = 2t_2 + 4t_3 \\
 & & = 2(-1) + 4(1) & = 2(1) + 4(2) \\
 & & = -2 + 4 & = 2 + 8 \\
 & & = 2 & = 10
 \end{array}$$

The first four terms are -1, 1, 2, and 10.

Example 2

Write a recursive formula for each sequence.

- (a) 2, 6, 18, 54, ... (b) 1, 2, 4, 7, 11, 16, ... (c) 4, 5, 20, 100, 2000, ...

Solution

- (a) To find any term, except the first one, multiply the previous term by 3.

$$t_1 = 2 \quad t_2 = 3t_1 \quad t_3 = 3t_2 \quad t_n = 3(t_{n-1})$$

- (b) To find the second term, add 1. To find the third term, add 2. Each number added is 1 less than the position of the term.

$$t_1 = 1 \quad t_2 = t_1 + 1 \quad t_3 = t_2 + 2 \quad t_n = t_{n-1} + (n - 1) \\ = t_1 + (2 - 1) \quad = t_2 + (3 - 1)$$

- (c) To find any term, except for the first two terms, multiply the two previous terms.

$$t_1 = 4 \quad t_2 = 5 \quad t_3 = t_2 \times t_1 \quad t_4 = t_3 \times t_2 \quad t_n = (t_{n-1})(t_{n-2})$$

Example 3

Todd fractured his wrist while he was Rollerblading. To relieve the pain, his doctor suggested that he take 500 mg of aspirin every 6 h. At this rate, only 26% of the aspirin remains in his body by the time he takes a new dose.

- (a) Find the first six terms of the sequence that represents the amount of aspirin remaining in Todd's body after each dose of aspirin.
(b) Write a recursive formula for this sequence.
(c) Graph the sequence.
(d) Describe what happens to the amount of aspirin remaining in Todd's body as he takes more aspirin.

Solution

- (a) The initial dose, or first amount, is 500 mg. The second amount is 500 mg, with 26% of the first amount remaining in his body.

$$500 + 0.26(500) = 500 + 130 \\ = 630$$

The second amount of aspirin is 630 mg.

The third amount is 500 mg, with 26% of the second amount remaining in his body.

$$500 + 0.26(630) = 500 + 163.8 \\ = 663.8$$

Continue this pattern:

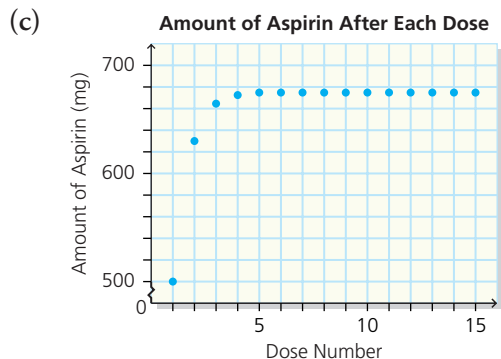
The fourth amount is $500 + 0.26(663.8) = 672.588$ mg.

The fifth amount is $500 + 0.26(672.588) = 674.872$ 88 mg.

The sixth amount is $500 + 0.26(674.87288) = 675.466$ 948 8 mg.

The first six terms of the sequence are 500, 630, 663.8, 672.588, 674.872 88, and 675.466 948 8.

(b) The recursive formula for this sequence is $t_1 = 500$, $t_n = 500 + 0.26(t_{n-1})$.



(d) It is apparent from the graph that, as Todd takes more aspirin, the amount that remains in his body increases and then becomes constant at a certain level. Create a spreadsheet to estimate the constant amount of aspirin remaining in Todd's body. Enter and format the information shown in cells A1, B1, and B2. Enter 1 to 10 in column A, beginning with A2, as shown.

	A	B
1	Dose Number	Amount of Aspirin (mg)
2	1	500
3	2	$=500+0.26*B2$
4	3	
5	4	
6	5	
7	6	
8	7	
9	8	
10	9	
11	10	

Select cells B3 to B11 and use the **Fill Down** command to complete the column.

	A	B
1	Dose Number	Amount of Aspirin (mg)
2	1	500
3	2	630
4	3	663.8
5	4	672.588
6	5	674.87288
7	6	675.4669488
8	7	675.6214067
9	8	675.6615657
10	9	675.6720071
11	10	675.6747218

The amount of aspirin remaining in Todd's body becomes constant at about 675.67 mg.

A

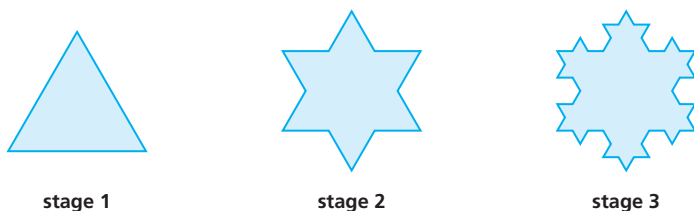
- Write the first five terms of each recursive sequence.
 - The first term is 3. Any other term is 9 more than the previous term.
 - The first term is 6. Any other term is the product of the previous term and 4.
 - The first term is 36. Any other term is one-half of the previous term.
 - The first term is 22. Any other term is 2 less than the previous term.
 - The first term is 3 and the second term is 4. Any other term is the product of the two previous terms.
 - The first term is -5 and the second term is 7. Any other term is the sum of the two previous terms.
- Write the first five terms of each sequence.
 - $t_1 = 4, t_n = t_{n-1} + 5$
 - $t_1 = -3, t_n = 2t_{n-1} - 2$
 - $t_1 = 8, t_n = t_{n-1} \div 2$
 - $t_1 = 4, t_n = (t_{n-1})^2$
 - $t_1 = 5, t_n = \frac{t_{n-1}}{n}$
 - $t_1 = 1, t_2 = 2, t_n = (t_{n-1})(t_{n-2})$
 - $t_1 = 2, t_2 = -1, t_n = (t_{n-1}) + (t_{n-2})$
 - $t_1 = 1, t_2 = 1, t_n = 2(t_{n-1}) - 3(t_{n-2})$
- Write a recursive formula for each sequence.
 - 2, 7, 12, 17, ...
 - $-2, 4, -8, 16, \dots$
 - 10, 5, 2.5, 1.25, ...
 - 2, 4, 16, 256, ...
 - 243, 81, 27, 9, 3, ...
 - 34, 25, 16, 7, ...
- Write a recursive formula for each sequence in question 1.
- A new boat depreciates by 15% of its original value each year.
 - What is the boat worth as a percent of the purchase price one year after it is bought?
 - The boat was bought for \$35 000. Write its value at the end of each year for the next five years as a sequence.
 - Write a recursive formula for this sequence.

B

- A farmer grows trees for a nursery. The farm has 4200 trees. Each year, the farmer harvests 20% of the trees and replants 850 trees.
 - Write a recursive formula for the total number of trees on the farm each year.
 - What happens to the total number of trees on the farm over time?



- 7. Knowledge and Understanding:** If $t_1 = 6$, $t_2 = 4$, $t_3 = 2$, and $t_n = (t_{n-3} + t_{n-2}) \times t_{n-1}$, find t_7 .
- 8.** Graph the first five terms of each recursive sequence.
- (a) $t_1 = 3$, $t_n = t_{n-1} + 2$ (b) $t_1 = -5$, $t_n = 2t_{n-1}$
 (c) $t_1 = 8$, $t_n = t_{n-1} \div 2$ (d) $t_1 = 2$, $t_n = 10 - t_{n-1}$
- 9.** Tarek applies for a job and is hired. His starting salary is \$40 000 for the first year. Each following year, his salary will be increased by \$1250.
- (a) Write a recursive formula for the sequence of Tarek's annual salaries.
 (b) Use the formula to find the first five terms of this sequence.
 (c) Find Tarek's salary ten years from now. Is it convenient to use the recursive formula in this situation? Explain.
- 10.** Here are the first three stages of a fractal called the Koch snowflake. To create the snowflake, start with an equilateral triangle. For stage 2, begin by dividing one side of the triangle into three and replace the middle segment with two lines that each have the same length as the middle segment. You can repeat these steps to create successive stages.



- (a) Each side of the original equilateral triangle is 18 cm. Copy and complete the table.

	Number of Segments	Length of Each Segment (cm)	Perimeter (cm)
stage 1	3	18	54
stage 2			
stage 3			

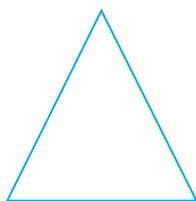
- (b) Write a recursive formula for each of the three sequences in the table in (a).
 (c) Find the number of segments in stage 6.
 (d) Find the length of each segment in stage 7.
 (e) What is the perimeter of the snowflake in stage 8?
- 11.** When a bungee jumper jumps off a platform, the bungee cord stretches 50 m. For each successive bounce, the cord stretches 80% of the previous length.
- (a) Write a recursive formula for the sequence of the length of the bungee chord after each bounce.

- (b) Determine the first five terms of this sequence.
- (c) Calculate the total distance travelled by the bungee jumper for the first five bounces.
- 12. Communication:** Create a sequence in which each term after the second term is a function of the previous two terms. Write a recursive formula for your sequence and use it to find the first ten terms.
- 13.** Nadia has a viral infection and must take 250 mg of medicine every 3 h. Only 30% of the medicine remains in her body by the time she takes the next dose. At what level (by mass) will the medicine remaining in her body become constant?
- 14.** Consider the sequence 2, 6, 12, 20, 30, ...
- (a) Find the next term in the sequence by using the pattern in the differences between consecutive pairs of terms.
- (b) Write a recursive formula for this sequence.
- 15. Application:** The fourth term of a sequence is 281. The recursive formula for the sequence is $t_n = 3(t_{n-1}) + 5$, for $n = 2, 3, \dots$. What is the first term of the sequence?
- 16.** In some lotteries, a million-dollar grand prize is paid in 20 annual payments of \$50 000 each. The winner receives the first \$50 000 on the day he or she claims the prize. The organizers of the lottery put just enough money into an account to pay the annual payments, and, after the 19th payment, the balance of the account is \$0. The account has an annual interest rate of 5%. Let a_0 represent the initial amount of money put into the account and a_n represent the amount of money in the account after the n th payment.
- (a) Write a recursive formula for a_n .
- (b) Use your calculator to find a_{19} , if $a_0 = \$604\,266.05$.
- (c) What is the advantage in paying the winner under these conditions?
- 17.** The population of a small town doubles every five years.
- (a) Today the population is 3500. Write a recursive formula for the sequence that represents the town's population every five years.
- (b) Determine the population 25 years from now.
- 18.** Every person has two parents, four grandparents, eight great-grandparents, and so on.
- (a) Write a recursive formula for the sequence of the number of ancestors going back each generation for any person.
- (b) How many ancestors does each person have going back for eight generations?
- 19.** If $t_1 = 2$, $t_2 = 4$, and $t_3 = 6$, find the next three terms of the sequence defined by $t_n = \frac{(t_{n-1})(t_{n-2})}{(t_{n-3})}$.

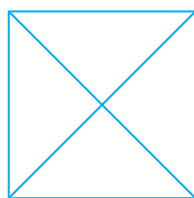
- 20. Thinking, Inquiry, Problem Solving:** In Ontario, many homes have filtration systems to remove impurities from the water. The water supplied to a particular home contains 4 mg of impurities, but the water is continuously recycled through a filtration system, and 52% of the impurities are removed each hour. How long will it take for the water to contain less than 0.02 mg of impurities?
- 21. Check Your Understanding:** If you were asked to find the 88th term of a sequence and were given both the general term of the sequence and a recursive formula, which would you use? Give an example to support your decision.



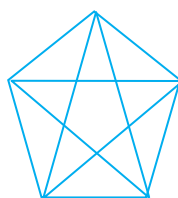
- 22.** Determine a recursive formula for the sequence 5, 9, 18, 34, 59, ...
- 23.** A theatre has 20 seats in the first row, 22 in the second row, 26 in the third row, 32 in the fourth row, and so on.
- Write a recursive formula for the sequence that represents the number of seats in each row, beginning with the first row.
 - The theatre has ten rows of seats. How many seats are in the last row?
 - How many seats are in the theatre?
- 24.** To draw a diagonal in a regular polygon, join a pair of non-adjacent vertices. How many diagonals does a decagon have altogether?



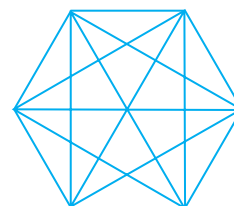
3 sides



4 sides



5 sides



6 sides



The Chapter Problem—Controlling Non-Native Plant Populations

In this section, you have studied recursive sequences. Apply what you have learned to answer these questions about the Chapter Problem on page 12.

- CP8.** Recall that the knapweed plant lives for five years and that the seeds of the plant are viable in soil for eight years. Use this information to extend the table on page 25 to include years 6 to 10.
- CP9.** Write a recursive formula for the sequence of the total number of plants in each year.
- CP10.** Write a recursive formula for the sequence of the number of seeds in the seedbank each year.