

Compound Interest: Amount and Present Value

1.8

Part 1: Investigating Compound Interest

Recall that in Part 2 of section 1.6, Juan's grandparents opened a simple interest savings account for him when he was born. On the same day, Juan's parents also open a savings account for him. Juan's father deposits \$10 000 in a savings account with an annual interest rate of 5%. The account pays **compound interest** on the balance at the end of each year.



At the end of the first year, interest is calculated on the principal and added to the balance in the account. The balance in the account at the beginning of the second period is the principal, plus the interest. At the end of the second year, interest is again calculated on the balance in the account, but, unlike simple interest, the interest is calculated on the principal, **plus** interest. In this way, the interest is compounded.

If there are no other deposits, what will be the balance in the account on Juan's 18th birthday?

Think, Do, Discuss

1. Calculate the interest earned at the end of the first year in the account opened by his parents.
2. Determine the balance in the account on Juan's first birthday.
3. Calculate the interest earned at the end of the second year.
4. Determine the balance in the account on Juan's second birthday.
5. Copy and complete the table on the next page. You could complete this table by hand, use a spreadsheet, or use the sequence editor of the TI-83 Plus calculator (**2nd** **LIST** **ENTER**).

Year	Balance at the Start of the Year (\$)	Interest (\$)	Balance at the End of the Year (\$)
0	10 000		
1			
2			
$\vdots$			
17			
18			

- What is the balance in the account on Juan's 18th birthday? How much of this amount is interest?
- Write the sequence that represents the balance in the account from the 1st birthday to the 18th birthday.
- Do the terms of this sequence increase by a constant value?
- Graph this sequence.
- Does the amount of money in the account grow at a linear or nonlinear rate? Explain.
- Using the sequence you found in step 7, divide each term by the previous term. What do you notice?
- What type of sequence have you created?
- Write each term of the sequence in step 7 as an expression of the value in step 11 and the original principal of \$10 000.
- Let n be a natural number that represents the number of years. Determine the general term of this sequence.
- Both Juan's parents and grandparents opened savings accounts, each with \$10 000. Explain why the balance in the compound interest account is much greater than the balance in the simple interest account on Juan's 18th birthday.
- If Juan left the money in the compound interest account until he retired at 65, what would be the balance in the account on his 65th birthday?
- Elena deposits \$5000 in an account that earns compound interest. The annual interest rate is 8%. Write the sequence that represents the yearly balance in the account each year for ten years. Determine the general term of this sequence.

Part 2: Defining Compound Interest—Amount

In Part 1 of this section, you calculated the interest for each year. Compound interest can also be calculated for these periods:

- **semiannually:** 2 times per year, at the end of every six months
- **quarterly:** 4 times per year, at the end of every three months
- **monthly:** 12 times per year, at the end of each month

Each of these periods, for example, six months, three months, or one month, is called a **conversion period**. The interest rate for each conversion period is a fraction of the annual interest rate, r .

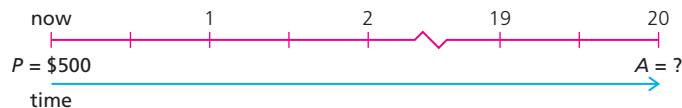
$$i = \frac{r}{\text{number of conversion periods per year}}$$

When solving a compound interest problem, draw a **time line** to organize information. A time line is simply a line divided into equal sections. Each section represents a conversion period. Julie invests \$500 for three years at 10%/a, compounded quarterly.



The left end point represents the present, “now,” and the right end point represents the end of the period, the end of the third year. The principal is on the left and the amount, to which the investment will grow, is on the right. There are four conversion periods in each year, and the time line has 12 sections.

For a long period, for example, 20 years, you need draw only part of the line, but include a break in the line, as shown. Joe invests \$1000 for 20 years at 10%/a, compounded semiannually. Here is the time line, where each year is divided into two equal sections and there is a break between year 2 and year 19.



Examining the Amount Under Different Conditions

You have inherited \$10 000 and have decided to invest the total inheritance in a savings account that pays 12%/a. You wish to invest the money for ten years, but you have four different options:

- **Option A:** Interest is compounded annually.
- **Option B:** Interest is compounded semiannually.
- **Option C:** Interest is compounded quarterly.
- **Option D:** Interest is compounded monthly.

Think, Do, Discuss

Steps 1 to 7 relate to option A.

1. How often is interest paid each year? Draw a time line.
2. How long is each conversion period?
3. What is the interest rate for each conversion period?
4. How many times will interest be paid for this investment?
5. Write the sequence that represents the balance in the account at the end of each conversion period.
6. By what factor does each new term increase? Determine the general term of this sequence.
7. Determine the balance in the account at the end of ten years.
8. Repeat steps 1 to 7 for options B, C, and D.
9. Graph each option on the same set of axes. Plot only the points that correspond to the balances in the account **at the end of each year** for each option.
10. Which option offers the best return on your initial investment? Explain your answer.

Part 3: Defining Compound Interest— Present Value

Marcus has just graduated from college, and he plans to buy a new car three years from now. He will need \$24 000 at that time. How much money must he invest today at 6%/a, compounded annually, to save the required amount?

Think, Do, Discuss

1. In this problem, which value is unknown, the future amount or the principal?
2. Draw a time line. Where should \$24 000 be placed on the time line?
3. Substitute the known values in the formula $A = P(1 + i)^n$ and solve for P .
4. What sum of money must Marcus invest today to have \$24 000 three years from now?
5. Express the formula $A = P(1 + i)^n$ in terms of P .
6. Use what you know about negative exponents to rewrite the formula for P so that n is negative.

Key Ideas

- In this section, interest is added or compounded to the principal before the interest for the next period is calculated. The formula for **compound interest** is

$$A = P(1 + i)^n$$

where A is the amount, the **future value** of an investment, or a loan,

P is the original principal invested or borrowed,

i is the interest rate per conversion period, and

n is the number of conversion periods.

- You can calculate compound interest in several ways:

annually	once per year	i = annual interest rate	n = number of years
semiannually	2 times per year	i = annual interest rate $\div$ 2	n = number of years $\times$ 2
quarterly	4 times per year	i = annual interest rate $\div$ 4	n = number of years $\times$ 4
monthly	12 times per year	i = annual interest rate $\div$ 12	n = number of years $\times$ 12

- The formula for compound interest is similar to the general term of a geometric sequence. Compare $A = P(1 + i)^n$ to $t_n = ar^{n-1}$.
- The amount that must be invested now, P , that will grow to a specific amount, A , in the future is called the **present value**. The formula for present value is

$$P = \frac{A}{(1 + i)^n} \text{ or } P = A(1 + i)^{-n}$$

where P is the present value,

A is the amount that the investment will grow to in the future,

i is the interest rate for each conversion period, and

n is the total number of conversion periods.

- A time line is useful for solving a compound interest problem, because you can use it to organize information and clarify whether the problem deals with present value or future value (amount).

Example 1

Martina invests \$5000 in a savings account that pays 5.25%/a, compounded annually. She does not make another deposit.

- (a) Create the geometric sequence of the year-end balances in the account.
- (b) Determine the amount in the account after 20 years.

Algebraic Solution

- (a) $r = 5.25\%$ or 0.0525 and $P = \$5000$

The amount at the end of the first year is

$$\begin{aligned}A &= P + I \\&= P + Prt \\&= P(1 + rt) \\A &= 5000(1 + 0.0525 \times 1) \\&= 5000(1.0525)^1 \\&= 5262.50\end{aligned}$$

This amount becomes the new principal at the beginning of the second year and earns interest at the same rate.

The amount at the end of the second year is

$$\begin{aligned}A &= 5262.50(1.0525) \text{ or } 5000(1.0525)^2 \\&= 5538.78\end{aligned}$$

The amount at the end of the third year is

$$\begin{aligned}A &= 5538.78(1.0525) \text{ or } 5000(1.0525)^3 \\&= 5829.57\end{aligned}$$

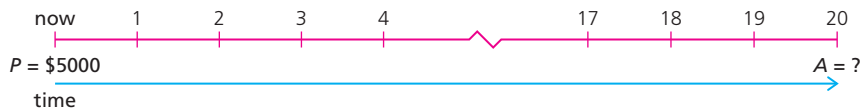
The general term is

$$t_n = 5000(1.0525)^n$$

The terms are $5000(1.0525)^1$, $5000(1.0525)^2$, $5000(1.0525)^3$, ...

Therefore, the sequence is 5262.50, 5538.78, 5829.57, ...

- (b) Draw a time line.



$A = P(1 + i)^n$, where $P = \$5000$, $i = 5.25\%$ or 0.0525 , and $n = 20$

$$\begin{aligned}A &= 5000(1 + 0.0525)^{20} \\&= 5000(1.0525)^{20} \\&= 13\,912.72\end{aligned}$$

At the end of the 20th year, there will be \$13 912.72 in the account.

Spreadsheet Solution

- (a) Enter the headings in cells A1 to D1 and enter the values in cells A2 and B2, as shown. Enter the formulas, as shown, in cells A3, B3, C3, and D3.

	A	B	C	D
1	Year	Balance at the start of the year	Interest	Balance at the end of the year
2	1	\$5,000	$=0.0525*B2$	$=B2+C2$
3	$=A2+1$	$=D2$	$=0.0525*B3$	$=B3+C3$
4				
5				

Select cells A3 to D5 and use **Fill Down**. Note: You may choose to format the cells differently.

	A	B	C	D
1	Year	Balance at the start of the year	Interest	Balance at the end of the year
2	1	\$5,000.00	\$262.50	\$5,262.50
3	2	\$5,262.50	\$276.28	\$5,538.78
4	3	\$5,538.78	\$290.79	\$5,829.57
5	4	\$5,829.57	\$306.05	\$6,135.62

The values in column B represent the terms of the sequence.
The sequence is 5262.50, 5538.78, 5829.57, ...

- (b) Expand the spreadsheet in (a).

Select cells A5 to D21 and **Fill Down**.

	A	B	C	D
1	Year	Balance at the start of the year	Interest	Balance at the end of the year
2	1	\$5,000.00	\$262.50	\$5,262.50
3	2	\$5,262.50	\$276.28	\$5,538.78
4	3	\$5,538.78	\$290.79	\$5,829.57
5	4	\$5,829.57	\$306.05	\$6,135.62
6	5	\$6,135.62	\$322.12	\$6,457.74
7	6	\$6,457.74	\$339.03	\$6,796.77
8	7	\$6,796.77	\$356.83	\$7,153.60
9	8	\$7,153.60	\$375.56	\$7,529.17
10	9	\$7,529.17	\$395.28	\$7,924.45
11	10	\$7,924.45	\$416.03	\$8,340.48
12	11	\$8,340.48	\$437.88	\$8,778.36
13	12	\$8,778.36	\$460.86	\$9,239.22
14	13	\$9,239.22	\$485.06	\$9,724.28
15	14	\$9,724.28	\$510.52	\$10,234.80
16	15	\$10,234.80	\$537.33	\$10,772.13
17	16	\$10,772.13	\$565.54	\$11,337.67
18	17	\$11,337.67	\$595.23	\$11,932.89
19	18	\$11,932.89	\$626.48	\$12,559.37
20	19	\$12,559.37	\$659.37	\$13,218.74
21	20	\$13,218.74	\$693.98	\$13,912.72

The value in cell D21 is the balance in the account at the end of the 20th year.
(This is also the value at the beginning of the 21st year.)

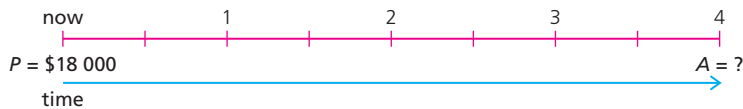
Example 2

Natalie invests \$18 000 at 8%/a, compounded semiannually.

- (a) Determine the value of the investment after four years.
- (b) Find the interest at this time.

Solution

- (a) The interest is compounded. Find the amount at the end of the fourth year.
Draw a time line.



$A = P(1 + i)^n$, where $P = \$18\,000$, $i = \frac{8\%}{2} = 4\%$ or 0.04 , and $n = 2 \times 4 = 8$

$$\begin{aligned} A &= 18\,000(1.04)^8 \\ &= 24\,634.24 \end{aligned}$$

At the end of the fourth year, the value of the investment is \$24 634.24.

- (b)
$$\begin{aligned} \text{interest} &= \text{amount} - \text{principal} \\ &= 24\,634.24 - \$18\,000 \\ &= 6634.24 \end{aligned}$$

The interest is \$6634.24.

Example 3

Alwynn invests \$500 in an account that earns 6%/a, compounded monthly.

Peter also invests \$500 at the same time, but in a different account that earns 6%/a, simple interest.

- (a) Determine the difference between their investments at the end of the fifth year.
- (b) Using graphing technology, compare the balance in each account at the end of each month for ten years.
- (c) Discuss how the two investments are different.

Solution

- (a) For Alwynn's investment, $A = P(1 + i)^n$, where $P = \$500$, $i = \frac{6\%}{12} = 0.5\%$ or 0.005 , and $n = 5 \times 12 = 60$.

$$\begin{aligned} A &= 500(1.005)^{60} \\ &= 674.43 \end{aligned}$$

For Peter's investment, $A = P + Prt$, where $P = \$500$, $r = 6\%$ or 0.06 , and $t = 5$

$$\begin{aligned} A &= \$500 + 500(0.06)(5) \\ &= \$650 \end{aligned}$$

Alwynn's investment is worth more after five years.

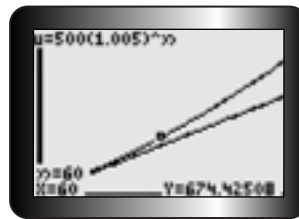
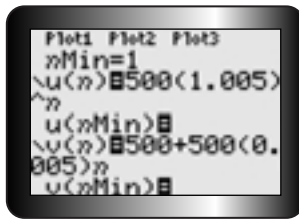
The difference is $\$674.43 - \$650$ or $\$24.43$.

- (b) To graph the monthly balances, determine the interest rate for each month, which is, for both cases, 0.005 .

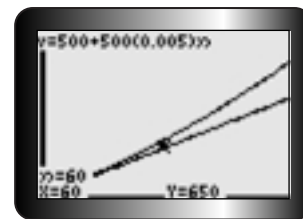
For Alwynn's investment, graph $t_n = 500(1.005)^n$.

For Peter's investment, graph $t_n = 500 + 500(0.005)n$.

In both cases, n is a natural number from 1 to 120 (120 months = 10 years).



Alwynn's account



Peter's account

- (c) From the graphs, Alwynn's investment is growing at a much faster rate than Peter's. As time passes, the difference between the monthly balances will increase. Peter's monthly balance grows at a constant rate, while Alwynn's monthly balance grows by increasing amounts from month to month.

Example 4

Determine the present value of an investment that will be worth \$5000 in ten years.

The interest rate is $4.8\%/a$, compounded quarterly.

Solution

Draw a time line.



$P = A(1 + i)^{-n}$, where $A = \$5000$, $i = \frac{4.8\%}{4} = 1.2\%$ or 0.012 , and $n = 10 \times 4 = 40$

$$\begin{aligned} P &= \$5000(1.012)^{-40} \\ &= \$5000(0.620\ 553\ 886\ 4) \\ &= \$3102.77 \end{aligned}$$

Note: To calculate $(1.012)^{-40}$ using the TI-83 Plus calculator, enter

$\boxed{1} \boxed{.} \boxed{1} \boxed{0} \boxed{1} \boxed{2} \boxed{\wedge} \boxed{(-)} \boxed{4} \boxed{0}$. For most scientific calculators, enter

$\boxed{1} \boxed{.} \boxed{1} \boxed{0} \boxed{1} \boxed{2} \boxed{y^x} \boxed{+/-} \boxed{4} \boxed{0} \boxed{=}$.

Practise, Apply, Solve 1.8

A

1. For each situation, determine
 - i. the interest rate for each conversion period
 - ii. the number of conversion periods
 - (a) an investment at 6%/a, compounded annually, for 5 years
 - (b) a loan at 8%/a, compounded semiannually, for 9 years
 - (c) a deposit at 4.4%/a, compounded quarterly, for 7 years
 - (d) a loan at 6%/a, compounded monthly, for 3 years
 - (e) an investment at 5%/a, compounded annually, for 5 years
 - (f) a deposit at 8.2%/a, compounded quarterly, for 16 months
 - (g) an investment at 4.5%/a, compounded semiannually, for 42 months
 - (h) a loan at 5.2%/a, compounded weekly, for 2 years
 - (i) a deposit at 5.475%, compounded daily, for 3 years
2. Evaluate to four decimal places.
 - (a) $(1.1)^{12}$
 - (b) $(1.05)^{-20}$
 - (c) $(1.025)^{36}$
 - (d) $(1.002)^{-45}$
 - (e) $200(1.04)^{24}$
 - (f) $5000(1.12)^{-5}$
 - (g) $6525(1.03)^{-22}$
 - (h) $10\,000(1.006)^{36}$
3. In the formula $A = P(1 + i)^n$, what does each variable represent?
 - (a) A
 - (b) P
 - (c) i
 - (d) n
4. For each situation, determine
 - i. the amount
 - ii. the interest earned
 - (a) \$4000 borrowed for 4 years at 3%/a, compounded annually
 - (b) \$7500 invested for 6 years at 6%/a, compounded monthly
 - (c) \$15 000 borrowed for 5 years at 2.4%/a, compounded quarterly
 - (d) \$28 200 invested for 10 years at 5.5%/a, compounded semiannually
 - (e) \$850 financed for 1 year at 3.65%/a, compounded daily
 - (f) \$2225 invested for 47 weeks at 5.2%/a, compounded weekly
5. For each situation, determine
 - i. the present value of
 - ii. the interest earned on
 - (a) an investment that will be worth \$5000 in 3 years. The interest rate is 4%/a, compounded annually.
 - (b) an investment that will be worth \$13 500 in 4 years. The interest rate is 6%/a, compounded monthly.
 - (c) a loan of \$11 200 due in 5 years, with interest of 4.4%/a, compounded quarterly.

- (d) an investment that will be worth \$128 500 in 8 years. The interest rate is 6.5%/a, compounded semiannually.
- (e) a loan of \$850 due in 400 days, with interest of 5.84%/a, compounded daily.
- (f) an investment that will be worth \$6225 in 100 weeks. The interest rate is 13%/a, compounded weekly.

6. **Communication:** Explain the difference between the amount of an investment and its present value. Provide an example.

B

7. Margaret can finance the purchase of a new \$949.99 refrigerator in two ways:
- **Plan A:** no money down, finance at 5%/a for 2 years
 - **Plan B:** no money down, finance at 5%/a, compounded quarterly for 2 years
- Which plan should she choose? Justify your answer.
8. Find the balance of the investment if \$1000 is compounded annually, at 5%/a, for
- (a) 10 years.
 - (b) 20 years.
 - (c) 30 years.
9. About when will the amount with compound interest in question 8 be twice the original principal?
10. On the day his son is born, Mike wishes to invest a single sum of money that will grow to \$10 000 when his son turns 21. If Mike invests the money at 4%/a, compounded semiannually, how much must he invest today?
11. **Knowledge and Understanding:** How much will \$7500 be worth if it is invested now for 10 years at 6%/a, compounded annually? Verify your calculation by determining the present value of the investment.
12. Jeannie bought a \$1000 Canada Savings Bond at work. Each month, \$83.33 is deducted from her monthly paycheque to finance the bond. The bond pays 5%/a, compounded annually, and matures seven years from the date of the first payment. Determine the value of the bond.
13. Barry bought a boat two years ago and at that time paid a down payment of \$10 000 cash. Today he must make the second and final payment of \$7500, which includes the interest charge on the balance owing. Barry financed this purchase at 6.2%/a, compounded semiannually. Determine the purchase price of the boat.
14. Tiffany deposits \$9000 in an account that pays 10%/a, compounded quarterly. After three years, the interest rate changes to 9%/a, compounded semiannually. Calculate the value of her investment two years after this change.
15. Exactly six months ago, Lee borrowed \$2000 at 9%/a, compounded semiannually. Today he paid \$800, which includes principal and interest. What must he pay to close the debt at the end of the year (six months from now)?

16. Today Sigrid has \$7424.83 in her bank account. For the last two years, her account has paid 6%/a, compounded monthly. Before then, her account paid 6%/a, compounded semiannually, for four years. If she made only one deposit six years ago, determine the original principal.
17. Explain, with an example, why compounded interest is comparable to a geometric sequence.
18. **Application:** On June 1, 1996, Anna invested \$2000 in a money market fund that paid 6%/a, compounded monthly. After five years, her financial advisor moved the accumulated amount to a new account that paid 8%/a, compounded quarterly. Determine the balance in her account on January 1, 2008.
19. Bernie deposited \$4000 into the “Accumulator Account” at his bank. During the first year, the account pays 4%/a, compounded quarterly. As an incentive to the bank’s customers, this account’s interest rate is increased by 0.2% each year. Calculate the balance in Bernie’s account after three years.
20. On the day Sarah was born, her grandparents deposited \$500 in a savings account that earns 4.8%/a, compounded monthly. They deposited the same amount on her 5th, 10th, and 15th birthdays. Determine the balance in the account on Sarah’s 18th birthday.
21. **Thinking, Inquiry, Problem Solving:** On the first day of every month, Josh deposits \$100 in an account that pays 4%/a, compounded monthly.
- (a) Determine the balance in his account after one year.
 - (b) What single sum of money invested today, at the same interest rate, will accumulate to the same amount under Josh’s original investment scheme?
22. **Check Your Understanding:** Matt invests \$500 at 8%/a, compounded semiannually. On the same day, Justin invests \$500 at 8%/a, compounded quarterly. Who will have more money after five years? Explain, including appropriate calculations.
- C**
23. Determine the interest rate that would cause an investment to double in seven years if interest is compounded annually.
24. If \$500 was invested at 8%/a, compounded annually, in one account, and \$600 was invested at 6%/a, compounded annually, in another account, then when would the amounts in both accounts be equal?
25. For compound interest, does doubling the interest rate cause the amount to double? Explain.