

1.11 Solving Exponential Equations

In an exponential equation, at least one of the exponents is, or has, a variable. Some examples of exponential equations are $2^x = 8$, $5^{x-3} = 10^{2x}$, and $5210 = 4000(1.05)^n$.

Part 1: Approximating a Solution

Martin invested \$1000 at 8%/a, compounded annually. Today the investment is worth \$4660.96. For how long has the money been invested?



Think, Do, Discuss

1. Use the formula for compound interest to write an equation for this situation.
2. How would you describe this equation?
3. Use the expression on the right side of the equal sign to complete the table.

Years (n)	0	2	4	6	8	10	12	14
Amount (\$)								

4. Using the information in the table, graph this relationship. In this case, can you draw a **curve of best fit** through the points? Explain.
5. Can you use the graph in step 4 to solve the problem? What must you do with the graph so that you may solve the problem?
6. Adjust the graph accordingly and determine for how long the money has been invested.
7. Are you confident that your answer is accurate? Explain.
8. Using the exponent key on your calculator, guess and check to solve the equation in step 1.
9. Which method, graphing or using a calculator, gives an exact solution? Explain.

Part 2: Exponential Equations with the Same Base

An Algebraic Solution

You can use algebra to determine an exact solution for an exponential equation when the powers on each side of the equation have the same base.

In the equation $a^x = a^y$, the bases are the same. For this equation to be true, the exponents must be equal. That is, if $a^x = a^y$, then $x = y$.

Example 1

Solve.

(a) $3^{2x} = 81$

(b) $5^{2x-1} = \frac{1}{125}$

(c) $36^{2x+4} = \sqrt{1296^x}$

Solution

(a) $3^{2x} = 81$
 $3^{2x} = 3^4$
 $\therefore 2x = 4$
 $x = 2$

Express 81 as a power of 3.
 The same bases must have equal exponents.
 Solve for x .

Verify that $x = 2$.

L.S.	R.S.
3^{2x}	81
$= 3^{2(2)}$	
$= 3^4$	
$= 81$	

Since L.S. = R.S.,
 the solution is correct.

(b) $5^{2x-1} = \frac{1}{125}$
 $5^{2x-1} = 5^{-3}$
 $\therefore 2x - 1 = -3$
 $2x = -3 + 1$
 $2x = -2$
 $x = -1$

Express $\frac{1}{125}$ as a power of 5.
 The same bases must have equal exponents.
 Solve for x .

(c) $36^{2x+4} = \sqrt{1296^x}$

Express the left side as a power of 6 and the right side using a rational exponent.

$$(6^2)^{2x+4} = 1296^{\frac{x}{2}}$$

Express the right side as a power of 6.

$$6^{2(2x+4)} = (6^4)^{\frac{x}{2}}$$

Apply the power of a power law and multiply the exponents.

$$\begin{aligned} 6^{4x+8} &= 6^{2x} \\ \therefore 4x + 8 &= 2x \\ 4x - 2x &= -8 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

The same bases must have equal exponents.
 Solve for x .

Solving Exponential Equations Involving Quadratic Equations

Some exponential equations can be more difficult to solve. In the following examples, quadratic equations must be used to solve the original exponential equation.

Example 2

Solve.

(a) $2^{x^2 + 2x} = \frac{1}{2}$ (b) $2^{2x} - 2^x = 12$

Solution

(a) $2^{x^2 + 2x} = \frac{1}{2}$

$$2^{x^2 + 2x} = 2^{-1}$$

$$\therefore x^2 + 2x = -1$$

$$x^2 + 2x + 1 = 0$$

$$(x + 1)(x + 1) = 0$$

$$x + 1 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = -1 \quad \text{or} \quad x = -1$$

Express the right side as a power of 2.

The same bases must have equal exponents.

Rearrange the quadratic equation so the right side is 0.

Factor the left side.

Set each factor equal to zero.

(b) $2^{2x} - 2^x = 12$

$$(2^x)^2 - 2^x = 12$$

$$(2^x)^2 - 2^x - 12 = 0$$

$$a^2 - a - 12 = 0$$

$$(a - 4)(a + 3) = 0$$

$$a - 4 = 0 \quad \text{or} \quad a + 3 = 0$$

$$a = 4 \quad \text{or} \quad a = -3$$

$$2^x = 4 \quad \text{or} \quad 2^x = -3$$

Apply the power of a power law.

Rearrange the quadratic equation so that the right side is 0.

Let $a = 2^x$.

Factor the left side.

Solve each factor.

Substitute 2^x for a .

Solve $2^x = 4$. The equation $2^x = -3$ has no solution, since any positive base raised to any exponent results in a positive value.

$$2^x = 2^2$$

$$\therefore x = 2$$

Finding the Number of Terms in a Geometric Sequence

Use an exponential equation to determine the number of terms in any finite geometric sequence.

Example 3

Determine the number of terms in the sequence 4, 20, 100, ..., 1 562 500.

Solution

This sequence is geometric, with $a = 4$ and $r = 5$.

$$\begin{aligned}
 t_n &= ar^{n-1} \\
 t_n &= 4(5)^{n-1} && \text{Let } t_n = 1\,562\,500. \\
 1\,562\,500 &= 4(5)^{n-1} && \text{Divide both sides by 4.} \\
 \frac{1\,562\,500}{4} &= \frac{4(5)^{n-1}}{4} \\
 390\,625 &= 5^{n-1} && \text{Express 390 625 as a power of 5.} \\
 5^8 &= 5^{n-1} \\
 \therefore 8 &= n - 1 && \text{Solve for } n. \\
 8 + 1 &= n \\
 9 &= n
 \end{aligned}$$

The sequence has nine terms.

Exponential Growth and Decay Problems

Exponential growth and radioactive decay (half-life) involve exponential equations.

Example 4

Radioactive material decays according to the relationship $M = c\left(\frac{1}{2}\right)^{\frac{t}{b}}$, where

M is the mass of the decayed radioactive material, in grams,

c is the original mass of the radioactive material, in grams,

$\frac{1}{2}$ is the decay factor,

t is the time, as specified, and

b is the half-life of the radioactive material.

Radon has a half-life of 25 days. How much time does it take for a sample of 200 g to decay to 12.5 g?

Solution

$M = c\left(\frac{1}{2}\right)^{\frac{t}{b}}$, and $M = 12.5$ g, $c = 200$ g, and $b = 25$ days

$$12.5 = 200\left(\frac{1}{2}\right)^{\frac{t}{25}} \quad \text{Divide both sides by 200.}$$

$$\frac{12.5}{200} = \frac{200\left(\frac{1}{2}\right)^{\frac{t}{25}}}{200} \quad \text{Simplify.}$$

$$\frac{1}{16} = \left(\frac{1}{2}\right)^{\frac{t}{25}} \quad \text{Express each side as a power of 2.}$$

$$2^{-4} = (2^{-1})^{\frac{t}{25}} \quad \text{Apply the power of a power law to the right side and multiply the exponents.}$$

$$2^{-4} = 2^{-\frac{t}{25}} \quad \text{The same bases must have equal exponents.}$$

$$-4 = -\frac{t}{25} \quad \text{Solve for } t.$$

$$25(-4) = 25\left(-\frac{t}{25}\right)$$

$$-100 = -t$$

$$100 = t$$

The radon decays to 12.5 g in 100 days.

Part 3: Solving an Exponential Equation Using Graphing Technology

When you cannot find an exact solution, use graphing technology to find a very close approximation.

Example 5

A bacteria culture doubles in size every 30 min. If the initial population of the culture is 10 bacteria, how long will the population take to reach 1 000 000?

Solution

The sequence 10, 20, 40, 80, ... is geometric.

$$t_n = ar^{n-1}$$

$$t_n = 10(2)^{n-1}$$

$$1\,000\,000 = 10(2)^{n-1}$$

Substitute the values of a and r .

Substitute 1 000 000 for t_n .

You could graph $y = 10(2)^{n-1}$ and then

TRACE to find the value of n . But use the zero operation to get a more accurate answer.

First divide both sides by 10 and rearrange the equation so that one side is 0.

$$100\,000 = (2)^{n-1}$$

Subtract 100 000 from both sides.

$$0 = (2)^{n-1} - 100\,000$$

You must find the zero of the graph of

$y = (2)^{n-1} - 100\,000$ to find the solution.

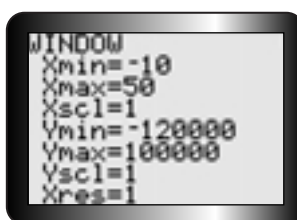
Graph $y = (2)^{n-1} - 100\,000$ and use the graph to determine the value of n when $y = 0$, that is, determine the zero. For the TI-83 Plus calculator, ensure that the selected graphing mode is **function** (**MODE**).

1. Enter the equation.



step 1

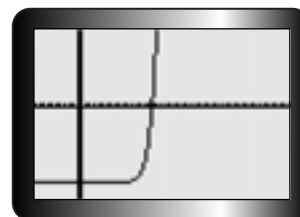
2. Adjust the window.



step 2

3. Draw the graph.

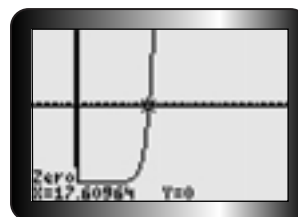
Press **GRAPH**.



step 3

4. Determine the zero using the zero operation.

Press **2nd** **TRACE** **2** **ENTER**. Respond to **Left Bound?** by moving the cursor to any point along the curve below the x -axis. Respond to **Right Bound?** by moving the cursor to any point along the curve above the x -axis. Respond to **Guess?** by pressing **ENTER**.



step 4

The population of the culture grows to 1 000 000 between the 17th and 18th terms of the sequence. Since the time between terms, the doubling period, is 30 min, then the time taken to reach 1 000 000 is

$$17.609\,64 \times 30 = 528.2892$$

The population of the bacteria colony takes 528.2892 min or 8 h 48 min 17 s to grow to 1 000 000.

Consolidate Your Understanding

1. Write an example of an exponential equation. Explain how an exponential equation is different from
 - (a) a linear equation
 - (b) a quadratic equation
2. What must be true of the expressions on both sides of an exponential equation to solve it algebraically?
3. How can you verify your solutions for an exponential equation?
4. Do all exponential equations have exact solutions? Explain.

Focus 1.11

Key Ideas

- In an **exponential equation**, at least one of the exponents is, or has, a variable. For example, $3^{4x+2} = 27^{x-2}$ is an exponential equation.
- You can determine an approximate solution to any exponential equation by graphing and then by interpolating or extrapolating.
- You can also determine an approximate solution to any exponential equation by using a scientific or graphing calculator to guess and check.
- If both sides of an exponential equation have the same base, you can solve this exponential equation algebraically. The algebraic solution in this case is exact. If the bases are the same, then the exponents are equal. That is, if $m^x = n^y$ and $m = n$, then $x = y$.
- If you use graphing technology to create a graph, then the solutions to the equation tend to be more accurate than the solutions found from a hand-drawn graph.
- You should verify the solutions to an exponential equation by substitution.

Practise, Apply, Solve 1.11

A

1. Express each as a power of 3.

(a) 27 (b) 81 (c) $\frac{1}{9}$ (d) 9^{2x} (e) $\left(\frac{1}{27}\right)^x$

2. Determine which value of x is the solution to the equation.

(a) $3^{2x-5} = 27$ (b) $4^{x+3} = 64$ (c) $5^{x+2} = \frac{1}{25}$ (d) $2^{5x+2} = \sqrt{2}$
 i. $x = 1$ i. $x = 1$ i. $x = 0$ i. $x = -\frac{3}{10}$
 ii. $x = 4$ ii. $x = 0$ ii. $x = -4$ ii. $x = -\frac{1}{2}$

3. Determine the exact solutions algebraically.

(a) $2^x = 27$ (b) $5^x = 5^3$ (c) $3^{x+6} = 3^{12}$ (d) $10^{2x-1} = 10^3$
 (e) $2^{2x-1} = 2^{x+9}$ (f) $7^{3x+2} = 7^{2x+5}$ (g) $4^{2x} = 4^8$ (h) $5^x = 5^{3x-12}$

4. Find the exact roots of each equation.

(a) $2^x = 32$ (b) $3^x = 27$ (c) $3^x = 9^{x-1}$ (d) $5^x = 3125$
 (e) $4(2^x) = 32$ (f) $5^x = \frac{1}{125}$ (g) $6^x = \sqrt[3]{6}$ (h) $3^{-x} = \frac{1}{81}$

5. Determine the approximate solutions using your calculator and guessing and checking. Round to two decimal places.

(a) $2^x = 20$ (b) $5^x = 35$ (c) $3^x = 100$ (d) $10^x = 800$
 (e) $2^{2x} = 50$ (f) $7^{3x} = 150$ (g) $4^{2x-1} = 80$ (h) $200 = 5^{3x+2}$

B

For questions 6 to 9, determine the exact solutions algebraically.

6. Solve each equation without using a calculator.

(a) $4^x = 8\sqrt{2}$ (b) $3^x = \sqrt[5]{9}$ (c) $125^x = 25\sqrt{5}$ (d) $8x = 16\sqrt[3]{2}$

7. (a) Solve. $\left(\frac{1}{9}\right)^{x+2} = \left(\frac{1}{27}\right)^{x+3}$

(b) Verify your answer in (a).

8. Solve each equation.

(a) $2^{7-x} = \frac{1}{2}$ (b) $2^{x-2} = 4^{x+2}$ (c) $\left(\frac{1}{4}\right)^{x-2} = \left(\frac{1}{8}\right)^{x+1}$
 (d) $9^{2x+1} = 81(27^x)$ (e) $2^{2x+2} + 7 = 71$ (f) $(2^{x+1})(4^{x+1})(8^{x+1}) = 128^x$

9. Determine the solution or solutions of each equation.

(a) $2^{x^2} = 32(2^{4x})$ (b) $3^{x^2} = 27(3^{2x})$ (c) $9^{x+2} = \left(\frac{1}{27}\right)^{x+2}$
 (d) $2^{r^2+6r} = 2^{-8}$ (e) $3^{x^2+20} = \left(\frac{1}{27}\right)^{3x}$ (f) $(8)^{x^2} = (4)^{4-5x}$
 (g) $(2^{x-4})^x = 32$ (h) $9^{x^2+1} = (27^x)(3^{2x})$ (i) $2^{x^2} = (16^{x-1})(2^x)$

- 10.** Determine the number of terms in each geometric sequence.
- (a) 2, 6, 18, ... , 39 366 (b) 5, 10, 20, ... , 10 240
- (c) -2, -10, -50, ... , -6250 (d) 16, 8, 4, ... , $\frac{1}{64}$
- (e) 3, 6, 12, ... , 384 (f) 4, 24, 144, ... , 186 624
- 11.** Determine the approximate solutions using graphing technology. Round to two decimal places.
- (a) $3^x = 30$ (b) $5^z = 10$ (c) $5.6^y = 60$ (d) $(1.04)^x = 2$
- (e) $25^{-x} = 10$ (f) $12^{2x} = 500$ (g) $4.1^{3x} = 40$ (h) $5^{2x-1} = 45$
- 12.** A bacteria culture doubles in size every 15 min. How long will it take for a culture of 20 bacteria to grow to a population of 163 840?
- 13. Knowledge and Understanding:** Determine the exact solution of $9^{2x+1} = 81(27^x)$.
- 14.** The use of wind turbines to generate electrical energy in Europe has increased exponentially. The energy produced by wind turbines between 1980 and 1995 can be modelled by the equation $y = 6.489(1.580)^x$, where x is the number of years since 1980 and y is the number of gigawatt-hours of energy produced.
- (a) Determine the amount of energy produced in 1980.
- (b) Determine the amount of energy produced in 1992.
- (c) In what year was 398.191 107 2 GW-h produced?
- (d) Determine when 500 GW-h of energy were produced.
- 15.** If \$500 is deposited in an account paying 8%/a, compounded semiannually, how long will it take for the deposit to increase to \$900?
- 16. Communication:** You can solve an exponential equation algebraically or by using a graph. Explain how you would decide which method to use. Include examples.
- 17.** Thorium-227 has a half-life of 18.4 days. How much time will a 50-mg sample take to decompose to 10 mg?
- 18. Thinking, Inquiry, Problem Solving:** When a plant or an animal dies, it stops absorbing carbon-14 from the atmosphere. Carbon-14 is an unstable radioactive isotope and decays over time. By measuring the amount of carbon-14 remaining in a sample from a plant or animal fossil, scientists can accurately predict the age of the specimen.
- (a) Research to find the half-life of carbon-14.
- (b) Estimate the age of the fossil of a leaf that contains 0.10% of the original amount of carbon-14.



19. If a sum of money is invested at $9\%/a$, compounded monthly, how much time will the investment take to double?
20. **Application:** Erica has a new job with a starting salary of \$32 000. She has been guaranteed an annual raise of 6% each year over the next ten years. When will her annual salary be more than \$50 000?
21. When a driver leaves the lights of a parked car turned on, the battery begins to discharge. A 12-V car battery loses about 8% of its charge each hour. The car must have at least 9 V of power to turn the starter's motor. How much time does the driver have to turn off the lights and still be able to start the car?
22. **Check Your Understanding:** Solve $31\,250 = 2(5)^{3x-12}$ algebraically. Verify your answer by finding the zero of the corresponding relationship using graphing technology.



23. Solve.
- (a) $2^{2x} - 33(2^x) + 32 = 0$ (b) $25^x - 30(5^x) + 125 = 0$
24. Solve. Round to two decimal places.
- (a) $9^{x^2-3x} = \frac{1}{27}$ (b) $(5^{x^2})(625) = \left(\frac{1}{125}\right)^{2x}$
25. Samantha invests \$1000 at $6\%/a$, compounded quarterly. Mark invests \$1500 at $5\%/a$, compounded quarterly. When will the balances in their accounts be equal?



The Chapter Problem—Controlling Non-Native Plant Populations

In this lesson, you have studied exponential equations. Use what you have learned to answer these questions about the Chapter Problem on page 12.

- CP19.** Use your model to predict the number of plants and seeds that exist after ten years.
- CP20.** How do the values in question CP19 compare with those in the table? Explain the similarities or differences.
- CP21.** Use your model, along with graphing technology, to determine when the number of plants exceeds 500 trillion.
- CP22.** If the site will support 105 mature knapweed plants per square metre, how many hectares would be affected after ten years?
(1 ha = 10 000 m²)