

In the previous section, you learned how to calculate the amount, or future value, of an ordinary simple annuity. The amount is the sum of the accumulated values of all the payments.

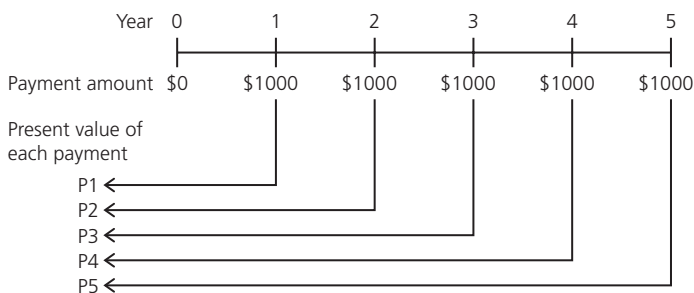
When people think about their retirement, they often hope to have a regular income from a pension or from an annuity. A **registered income fund** (RIF) is one type of annuity. This fund is a lump sum of money that is invested now so that a regular income may be drawn from the fund over a long period at some time in the future. For example, Blake wants to receive a \$1000 payment at the end of the next five years, so he invests an amount that is less than \$1000 today. The amount that he invests is less than the future value because the investment earns interest during the term.

The amount that is invested now to yield a larger amount in the future is called the **present value**. In this section, you will calculate the present value of an annuity by adding the present values of all the payments.

Part 1: Representing the Present Value of an Ordinary Simple Annuity with a Series

The St. Charles College Math Club wants to establish a math scholarship to reward its top graduating math student. The \$1000 scholarship will be awarded at the end of each year for the next five years. The club has some money to pay for the future scholarship payments. Calculate how much money is needed now by finding the present value of each of the five \$1000 awards and then adding the present values. The trust account pays 9%/a, compounded annually.

Here is the time line that shows the payments as present values.



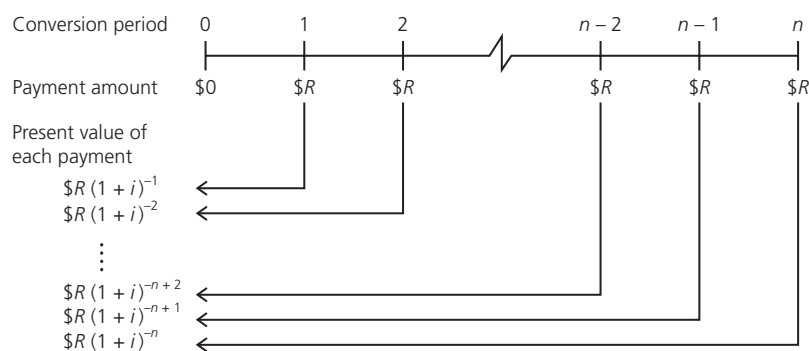
Think, Do, Discuss

1. Is \$1000 required now to cover the cost of the \$1000 scholarship that will be awarded at the end of the first year? Explain.

- The interest rate is $9\%/a$, compounded annually. Determine an expression that would represent the amount that is needed now to provide the first \$1000 scholarship payment at the end of the first year. Express the present value of the payment as a product rather than as a quotient by using negative exponents.
- Determine an expression for the present value of the second \$1000 scholarship. Express this present value as a product.
- Repeat step 3 to find expressions for the present values of the third, fourth, and fifth payments.
- Copy and complete the time line.
- Use the expressions you found above, *beginning with the present value of the fifth payment, P_5* , to write the series that represents the accumulated sum of all the present value payments. Do not find the value of each term.
- What is the first term? What do you multiply the first term by to get the second term? What is this type of series? What is the common ratio? How many terms does the series have?
- Use the formula you learned earlier in this chapter to find the sum of the present values of the payments. How much money would the club need to raise and then deposit in the trust account to sustain the scholarship over five years?

Part 2: Developing a Formula for the Present Value or Amount of an Ordinary Simple Annuity

Andrea wants to find an algebraic expression to represent the present, or discounted, value of an ordinary simple annuity. She lets R be the regular periodic payment of the annuity. She lets n be the total number of interest conversion periods, or the total number of payments. She drew the following time line.



Think, Do, Discuss

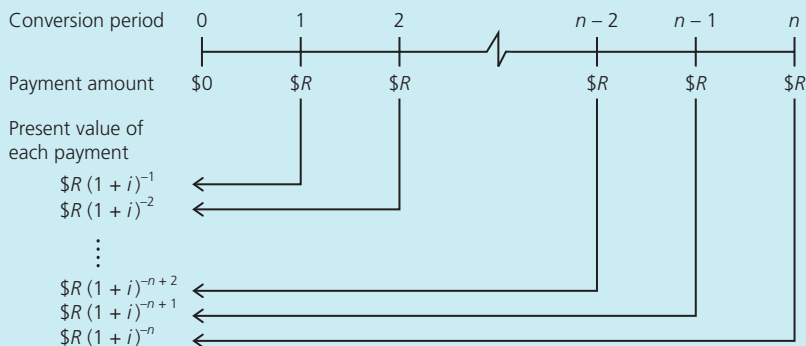
- What does i represent in the time line? Explain why the exponent in the present value of the first payment is -1 . Explain why the exponent in the present value of the last payment is $-n$.

- Given Andrea's time line, write the series of terms, PV , that represents the present or discounted value of each payment of the annuity, *beginning with the last payment*. What type of series is PV ?
- What is the first term of the series? What is the common ratio? Substitute these values in the formula you used in step 8 of Part 1.
- You can simplify the equation in step 3, but before you do, what do you notice about the bases of the powers in this equation? What do you do to the exponents when you multiply these powers? Use the distributive property to simplify your equation. The result is the present value, PV , of an ordinary simple annuity.
- A math scholarship awards \$500 at the end of each year for ten years. Use the formula you developed in step 4 to find the present value that should be deposited in a trust account that pays 11%/a, compounded annually, to sustain the scholarship.

Focus 2.8

Key Ideas

- The **present value**, or **discounted value**, of an annuity is the value at the beginning of the term of the annuity. The present value is the sum of all the present values of the payments.
- This time line represents the **present value** of an ordinary simple annuity.



- The present or discounted value of an ordinary simple annuity can be written as the geometric series

$$R(1+i)^{-n} + R(1+i)^{-n+1} + R(1+i)^{-n+2} + \dots + R(1+i)^{-2} + R(1+i)^{-1}$$

where R represents the regular payment

n represents the number of interest conversion periods or the total number of payments

i represents the interest rate per conversion period

- The present or discounted value of an ordinary simple annuity is the sum of a geometric series, $S_n = \frac{a(r^n - 1)}{r - 1}$, where $a = R(1 + i)^{-n}$, $r = 1 + i$, and n is the number of payments.
- Find the present or discounted value of an ordinary simple annuity, A or PV , using

$$A = PV$$

$$= R \times \frac{1 - (1 + i)^{-n}}{i}$$

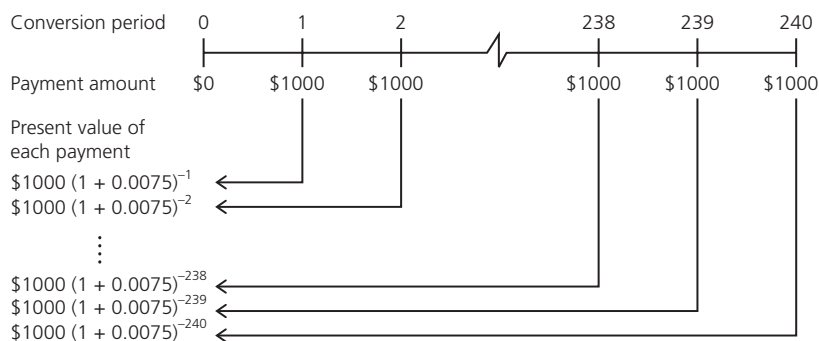
Example 1

Claire has just won a lottery. She is offered two prize options: \$125 000 cash today or payments of \$1000 each at the end of each month for 20 years. Claire expects a return of 9%/a, compounded monthly, if she invests the cash prize.

- Draw a time line to represent the present value of the payment option.
- Write the series that represents the annuity.
- Which option should Claire choose?

Solution

- The monthly interest rate is $0.75\% = 0.0075$ and the term is $12 \times 20 = 240$.



- The series is $S_{240} = 1000(1.0075)^{-240} + 1000(1.0075)^{-239} + 1000(1.0075)^{-238} + \dots + 1000(1.0075)^{-2} + 1000(1.0075)^{-1}$.

- Method 1:** Using $S_n = \frac{a(r^n - 1)}{r - 1}$

It is possible to misinterpret this problem by comparing the cash value today, \$125 000, to the total money paid under the payment option, $\$1000 \times 12 \times 20 = \$240\,000$.

The series is geometric, and $a = 1000(1.0075)^{-240}$, $r = 1.0075$, and $n = 240$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Substitute the values for a , r , and n .

$$S_{240} = 1000(1.0075)^{-240} \left[\frac{(1.0075)^{240} - 1}{1.0075 - 1} \right]$$

Simplify.

$$\begin{aligned} &\doteq 1000(0.166\ 412\ 844\ 8) (667.886\ 869\ 9) \\ &= 111\ 144.95 \end{aligned}$$

The present value of the payment option is \$111 144.95. Claire should choose the \$125 000 if she may invest it at 9%/a, compounded monthly.

Method 2: Using $PV = R \times \frac{1 - (1 + i)^{-n}}{i}$

The series represents the present or discounted value of an ordinary simple annuity where $R = 1000$, $i = 0.0075$, and $n = 240$.

$$PV = R \times \frac{1 - (1 + i)^{-n}}{i}$$

Substitute the values for R , i , and n .

$$= 1000 \times \frac{1 - (1.0075)^{-240}}{0.0075}$$

Simplify.

$$= 111\ 144.95$$

Using the formula for the present or discounted value also yields \$111 144.95.

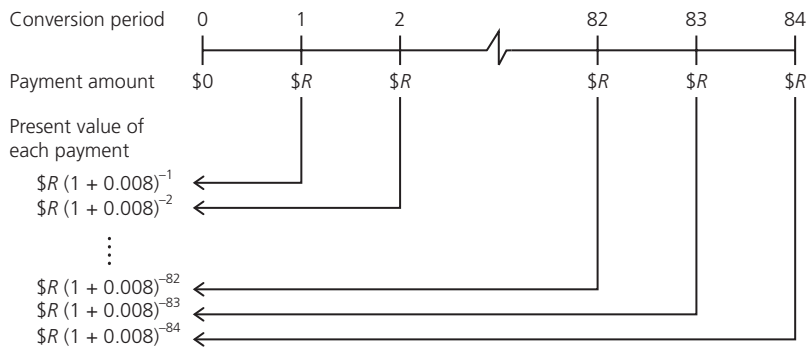
Example 2

The Sybulski family purchase a motor home. They borrow \$30 000 at 9.6%/a, compounded monthly, and they will make payments at the end of each month. They have two choices for the term: seven years or ten years.

- Draw a time line to represent the present value of the seven-year annuity.
- Write the series that represents the seven-year annuity.
- Find the monthly payment under each term.
- How much would they save in interest by selecting the shorter term?

Solution

- The monthly interest rate is $9.6\% \div 12 = 0.8\%$ or 0.008 and the number of periods is $12 \times 7 = 84$ for the seven-year term.



(b) The series is $S_{84} = R(1.008)^{-84} + R(1.008)^{-83} + R(1.008)^{-82} + \dots + R(1.008)^{-2} + R(1.008)^{-1}$, where $S_{84} = 30\,000$.

(c) Use $PV = R \times \frac{1 - (1 + i)^{-n}}{i}$. The present value of the loan is \$30 000. Find the monthly payment, R , under both terms.

For a seven-year term, $PV = 30\,000$, $i = 0.8\%$ or 0.008 , and $n = 84$.

$$\begin{aligned} PV &= R \times \frac{1 - (1 + i)^{-n}}{i} && \text{Substitute the values for } PV, i, \text{ and } n. \\ 30\,000 &= R \times \frac{1 - (1.008)^{-84}}{0.008} && \text{Simplify and solve for } R. \\ R &= 491.86 \end{aligned}$$

For a ten-year term, the values for PV and i are the same, but $n = 12 \times 10 = 120$.

$$\begin{aligned} PV &= R \times \frac{1 - (1 + i)^{-n}}{i} && \text{Substitute the values for } PV, i, \text{ and } n. \\ 30\,000 &= R \times \frac{1 - (1.008)^{-120}}{0.008} && \text{Simplify and solve for } R. \\ R &= 389.84 \end{aligned}$$

For a seven-year term, the payment would be \$491.86. For a ten-year term, the payment would be \$389.84.

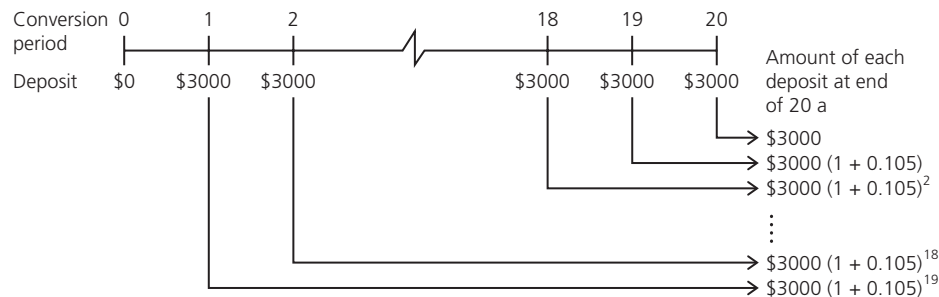
(d) The total repayment cost of the seven-year term is $\$491.86 \times 84 = \$41\,316.24$ and $\$389.84 \times 120 = \$46\,780.80$ for the ten-year term. The family would save \$5464.56 in interest by choosing the shorter term.

Example 3

Mr. Fabiilli deposits \$3000 in an RRSP each year. The interest rate for the RRSP is 10.5%/a, compounded annually. He contributes to this RRSP for 20 years, at the end of which time he retires. Upon retiring, Mr. Fabiilli decides to transfer this RRSP to a RIF. How much can he withdraw from the fund at the end of every three months if the fund earns 9%/a, compounded quarterly, and if he wishes to deplete the fund after 15 years? Use time lines to help you solve the problem.

Solution

First find the accumulated or future value of this RRSP in 20 years. Here is the time line, where $R = 3000$, $n = 20$, and $i = 10.5\%$ or 0.105 .



The accumulated or future value of the deposits is the geometric series,

$$S_{20} = 3000 + 3000(1.105) + 3000(1.105)^2 + \dots + 3000(1.105)^{18} + 3000(1.105)^{19}.$$

Use either $S_n = \frac{a(r^n - 1)}{r - 1}$, and $a = 3000$, $r = 1.105$, and $n = 20$ or

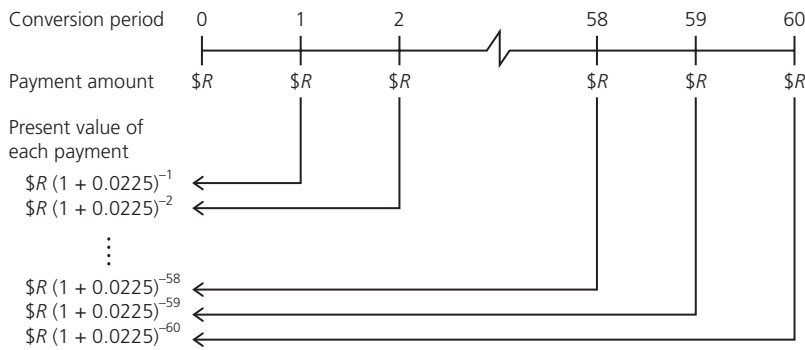
$FV = R \times \frac{(1 + i)^n - 1}{i}$, and $R = 3000$, $i = 0.105$, and $n = 20$ to find the accumulated value of his deposits.

$$\begin{aligned} S_{20} &= 3000 \times \frac{(1.105)^{20} - 1}{0.105} \\ &= 181\,892.42 \end{aligned}$$

The future value of the RRSP will be \$181 892.42.

Now find the quarterly payment that would yield the present value of \$181 892.42.

Here is the time line, where $n = 4 \times 15$ or 60, and $i = \frac{0.09}{4}$ or 0.0225.



The present value of the deposits is the geometric series $S_{60} = R(1.0225)^{-60} + R(1.0225)^{-59} + R(1.0225)^{-58} + \dots + R(1.0225)^{-2} + R(1.0225)^{-1}$, where $S_{60} = 181\,892.42$.

Solve for R .

Use the present value formula $PV = R \times \frac{1 - (1 + i)^{-n}}{i}$, where

$PV = 181\,892.42$, $n = 4 \times 15$ or 60, and $i = \frac{0.09}{4}$ or 0.0225.

$$\begin{aligned} 181\,892.42 &= R \times \frac{1 - (1.0225)^{-60}}{0.0225} && \text{Simplify and solve for } R. \\ R &= 5554.14 \end{aligned}$$

Therefore, Mr. Fabiilli can withdraw \$5554.14 from the fund at the end of every three months.

Practise, Apply, Solve 2.8

A

1. Evaluate to six decimal places.

(a) $(1.08)^{-10}$

(b) $(1.12)^{-6}$

(c) $(1.03)^{-24}$

(d) $(1.005)^{-60}$

(e) $(1.075)^{-20}$

(f) $(1.05)^{-48}$

2. Evaluate to two decimal places.

(a) $100(1.1)^{-24} \times \frac{(1.1)^{24} - 1}{1.1 - 1}$

(b) $900 \times \frac{1 - (1.075)^{-120}}{0.075}$

(c) $3000(1.05)^{-10} \times \frac{(1.05)^{10} - 1}{1.05 - 1}$

(d) $750 \times \frac{1 - (1.007)^{-240}}{0.007}$

3. Evaluate.

(a) $100(1.075)^{-24} + 100(1.075)^{-23} + 100(1.075)^{-22} + \dots + 100(1.075)^{-2} + 100(1.075)^{-1}$

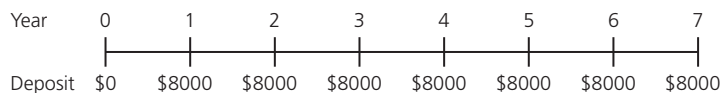
(b) $750(1.001)^{-240} + 750(1.001)^{-239} + 750(1.001)^{-238} + \dots + 750(1.001)^{-2} + 750(1.001)^{-1}$

(c) $1500(1.035)^{-10} + 1500(1.035)^{-9} + 1500(1.035)^{-8} + \dots + 1500(1.035)^{-2} + 500(1.035)^{-1}$

4. For each annuity

- find the present or discounted value of each payment at the beginning of the term
- write the present or discounted values of the payments as a series
- find the present or discounted value of the annuity

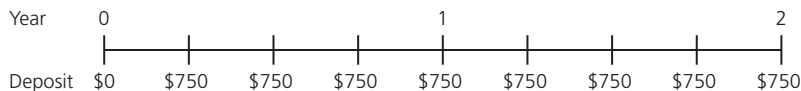
(a) The rate of interest is 9%/a, compounded annually.



(b) The rate of interest is 8%/a, compounded semiannually.



(c) The rate of interest is 8%/a, compounded quarterly.



5. (a) Draw a time line to represent the present value of this annuity: \$1200 is deposited at the *beginning* of every three months for three years in an account that pays 9%/a, compounded quarterly.
- (b) Describe this type of annuity.
- (c) Write the present values of all the payments as a series.
- (d) Find the present value of the annuity.

B

6. For each of the following
 - i. draw a time line to represent the present value of the annuity
 - ii. write the series that represents the present value of the annuity
 - iii. find the present value of the annuity
 - (a) \$8000 is deposited at the end of every year for 11 years at 7%/a, compounded annually
 - (b) \$650 is paid at the end of every 3 months for 6 years at 8%/a, compounded quarterly
 - (c) \$60 is deposited at the end of every week for 3 years at 13%/a, compounded weekly
 - (d) \$3800 is paid at the end of every 6 months for 8 years at 6.5%/a, compounded semiannually
7. The Ontario Association for Mathematics Education wishes to establish a math scholarship to reward the top mathematics educator in the province. The scholarship will be worth \$2000. The association will deposit a lump sum in a trust account that pays 7.5%/a, compounded annually, for ten years. Determine how much must be deposited now if the first award will be given
 - (a) one year from now
 - (b) immediately
 - (c) 4 years from now
 Include a time line and a series to represent each case.
8. Lily wants to buy a snowmobile. She can borrow \$7500 at 10%/a, compounded quarterly, if she repays the loan by making equal quarterly payments for four years.
 - (a) Draw a time line to represent the annuity.
 - (b) Write the series that represents the present value of the annuity.
 - (c) Find the quarterly payment that Lily must make.
9. Nader plans to buy a used car. He can afford to make payments of \$280 each month for a maximum of three years. The best interest rate he can find is 9.8%/a, compounded monthly. What is the most he can spend on a vehicle?



10. How much must Marie deposit now in a fund paying 6%/a, compounded semiannually, if she withdraws \$1000 every six months, starting six months from today, for the next ten years?
11. **Knowledge and Understanding:** Roxanne buys a DVD/CD player for \$50 down and ten monthly payments of \$40 each. The first payment is due next month.
 - (a) The interest rate is 18%/a, compounded monthly. What is the selling price of the player?
 - (b) What is the interest charge?
12. Shimon wants to purchase a speedboat that sells for \$22 000, including all taxes. The dealer offers either a \$2000 discount if Shimon pays the total amount in cash or a finance rate of 2.4%/a, compounded monthly, if Shimon makes equal monthly payments for five years.
 - (a) Determine the monthly payment that Shimon must make if he chooses the second offer.
 - (b) What is the total cost of the dealer's finance plan for the speedboat?
 - (c) To pay for the boat with cash now, Shimon knows that he can borrow the money from the bank at 6%/a, compounded monthly, over the same five-year period. Which offer should Shimon choose? Justify your answer.
13. René buys a computer system for \$80 down and 18 monthly payments of \$55 each. The first payment is due next month.
 - (a) The interest rate is 15%/a, compounded monthly. What is the selling price of the computer system?
 - (b) What is the finance charge?
14. The Peca family buy a cottage on Manitoulin Island for \$69 000. They plan to deposit \$5000 and then finance the remaining amount with a loan at 9%/a, compounded monthly. The loan payments are monthly. They may choose either a seven-year term or a ten-year term.
 - (a) Find the monthly payment required for each term.
 - (b) How much would they save in interest by selecting the shorter term?
 - (c) What other factors should the Pecas consider when making their financing decision?
15. **Application:** A car dealer advertises a new sports car. The selling price is \$32 000. The dealer offers this finance plan: the interest rate is 2.4%/a, compounded monthly, for five years with monthly payments. You can save \$3000 by paying cash for the sports car. Suppose you could obtain a loan from a financial institution at 5.4%/a, compounded monthly. What is the best way to buy the car? Show your work.

16. In Example 3, the interest rate for the RRSP is 10.5%/a and the interest rate for the RIF is 9%/a. Determine how much Mr. Fabiilli would be able to withdraw from the RIF at the end of every three months if the interest rate for the RRSP is 9%/a and the interest rate on his RIF is 7.5%/a.
17. **Communication:** In Example 3 of section 2.5, a spreadsheet was used to solve the following problem: “An ad in an electronics store window reads, ‘Big-screen television: Was \$3270: now \$2890 or \$500 down and \$122.81 for 24 months on approved credit.’ What is the annual interest rate, compounded monthly, of the store’s payment plan?” Discuss the merits and difficulties of trying to solve this problem using the present value of an annuity.
18. At the end of each year, Mr. Fox deposits \$2600 in an RRSP. The interest rate is 7.2%/a, compounded annually. He contributes to this RRSP for 20 years, at the end of which time he retires. Upon retiring, Mr. Fox decides to transfer all of this RRSP to an RIF. The interest rate for the RIF is 8.4%/a, compounded quarterly.
- (a) Mr. Fox wishes to deplete the fund after 15 years. Then what will be the quarterly payment from the RIF?
- (b) What would be the RIF payment if Mr. Fox had started contributing to his RRSP five years earlier, that is, if he had contributed to the RRSP for a total of 25 years?

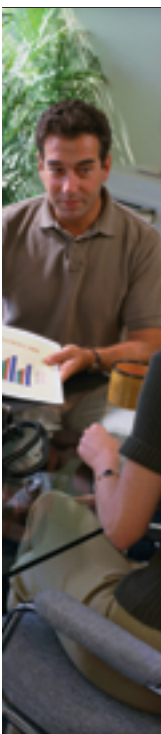
Use time lines to help you solve this problem.

19. A lottery offers winners two prize choices. Option A is \$1000 each week for life and Option B is \$660 000 in one lump sum. The current expected rate of return for a large investment is 6.76%/a, compounded weekly.
- (a) Which option would you suggest to a winner who expects to live for another 25 years?
- (b) At what point in time is Option A better than Option B?
20. **Check Your Understanding:** The present value of the last payment of an ordinary simple annuity is $2500(1.05)^{-36}$.
- (a) Describe two annuities, each with a different conversion period, that can be represented by the present value of this last payment.
- (b) Find the present values of all the payments for each annuity in (a).



21. **Thinking, Inquiry, Problem Solving:** Carissa claims that she has found a different method for finding the present value of an ordinary simple annuity. Instead of finding the present value of each payment, she finds the accumulated or future value of each payment. She then finds the sum of the future values of the payments. Finally, she finds the present value of this total sum.
- (a) Using Carissa’s method and referring to Example 1 of section 2.7, find the present value of the annuity where \$750 is deposited at the end of every three months in a savings account that pays 8%/a, compounded quarterly, for ten years.

- (b) Create another example to show that her claim is true. Include time lines.
- (c) In step 3 of the Think, Do, Discuss in Part 2, you developed the present value formula $\frac{R(1+i)^{-n}[(1+i)^n - 1]}{i}$. Use this formula to prove that Carissa's claim works for *all* simple ordinary annuities.
- 22.** Kyla has just graduated from university. She must repay her student loans that total \$17 000. She can afford to make monthly payments of \$325 each. The bank's interest rate is 7.2%/a, compounded monthly. Determine how long it will take Kyla to repay her student loan.
- 23.** Kumar wants to buy a used jeep that costs \$7500. He borrows the \$7500 at 11.4%/a, compounded monthly. Kumar decides that he can afford to pay \$280 each month toward the loan. How long will it take to Kumar to repay is loan? Use a spreadsheet to verify your answer.



The Chapter Problem—Financial Planning

In this section, you studied the present value of an annuity. Apply what you learned to answer these questions about the Chapter Problem on page 106.

- CP13.** Can you apply the formula for the present value of an ordinary simple annuity to determine Mr. Sacchetto's new monthly mortgage payment? Explain.
- CP14.** At age 55, Mr. Sacchetto may choose between two RIF options for investing the \$120 000 savings. Option A provides monthly payments from an RIF that pays 7.2%/a, compounded monthly. Option B provides monthly payments from an RIF that pays 7.25%/a, compounded quarterly.
- Can you use the formula for the present value of an ordinary simple annuity to solve for the RIF payment under each option? Explain.
 - For option A, draw a time line to represent the present value of the payments he would receive.
 - Determine the monthly RIF payment he would receive under option A.
- CP15.** Use the formula for the present value of an ordinary simple annuity to estimate the term of Mr. Sacchetto's car loan.