

In previous sections, you have worked with **ordinary simple annuities**. In an ordinary simple annuity, the payment interval corresponds to the interest conversion period. In this section, you will investigate annuities where this is not the case. An annuity where the payment interval does not correspond to the interest conversion period is called a **general annuity**.

When working through this section, remember to change the decimal mode on the TI-83 Plus calculator, as necessary.

Part 1: Equivalent Rates

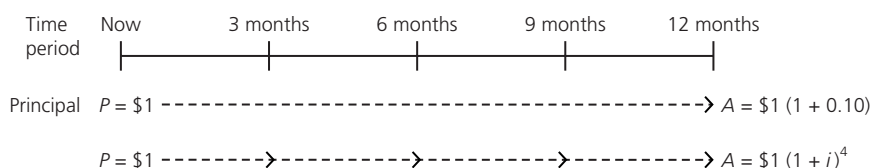
Two interest rates are said to be **equivalent** if they yield the same amounts at the end of one year, or at the end of any number of years. The **nominal** rate of interest is the annual rate.

Example 1

Suppose that \$1 is deposited in an account that earns 10%/a, compounded annually. Also suppose that another \$1 is deposited in another account that earns interest, but the interest is compounded quarterly. What is the interest rate of the second account so that the value in each account at the end of one year is the same?

Solution

Draw a time line to organize the given information.



Let i be the **quarterly** interest rate. Then $4 \times i$ is the *nominal* interest rate, compounded quarterly.

One dollar invested at 10%/a, compounded annually, will accumulate at the end of one year to $\$1(1 + 0.10)^1 = \$1.10$.

One dollar invested at $4i$ %/a, compounded quarterly, will accumulate at the end of one year to $\$1(1 + i)^4 = \$1(1 + i)^4$.

The accumulated values must be equal. Let $\$1(1 + i)^4$ equal $\$1.10$ and solve for i .

$$(1 + i)^4 = 1.10$$

Take the fourth root of each side.

$$1 + i = \sqrt[4]{1.10}$$

$$1 + i \doteq 1.024\ 113\ 689$$

Isolate i .

$$i = 0.024\ 113\ 689$$

Then $4 \times 0.024\,113\,689 \doteq 0.0965$.

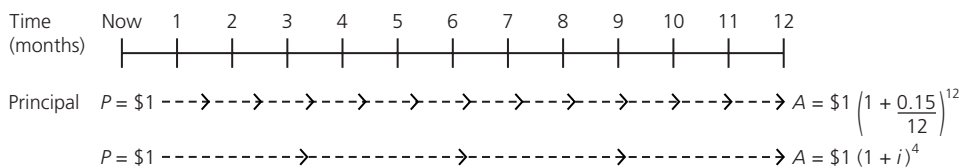
The interest rate that is equivalent to 10%/a, compounded annually, is 9.65%/a, compounded quarterly.

Example 2

What is the nominal interest rate, compounded quarterly, that is equivalent to 15%/a, compounded monthly?

Solution

Draw a time line to organize the information. Let i be the equivalent quarterly interest rate. Suppose again that \$1 is invested in two accounts at the beginning of the year.



Let i represent the quarterly rate.

At the end of one year, \$1, compounded quarterly, will accumulate to $(1 + i)^4$.

At the end of the same year, \$1, compounded monthly, at 15%/a, will accumulate to $\left(1 + \frac{0.15}{12}\right)^{12}$.

Let $(1 + i)^4$ equal $\left(1 + \frac{0.15}{12}\right)^{12}$ and solve for i .

$$(1 + i)^4 = \left(1 + \frac{0.15}{12}\right)^{12}$$

$$(1 + i)^4 = (1.0125)^{12}$$

Take the fourth root of each side.

$$1 + i = \sqrt[4]{(1.0125)^{12}}$$

$$1 + i = 1.0125^3$$

Isolate *i*.

$$i \doteq 0.037\ 970\ 703\ 1$$

Then $4 \times 0.037\,970\,703\,1 \doteq 0.151\,88$.

The interest rate per conversion period is about 3.797%. The nominal rate is $4 \times 0.037\,970\,703\,1$, or about 15.188%/a, compounded quarterly. This rate is equivalent to 15%/a, compounded monthly.

Part 2: Calculating the Future Value or the Present Value of an Ordinary General Annuity

In an ordinary general annuity, the payment interval does not coincide with the interest conversion period. To solve a problem involving an ordinary general annuity, you must first find the nominal interest rate that coincides with the payment interval. Then solve as if the problem were an ordinary simple annuity.

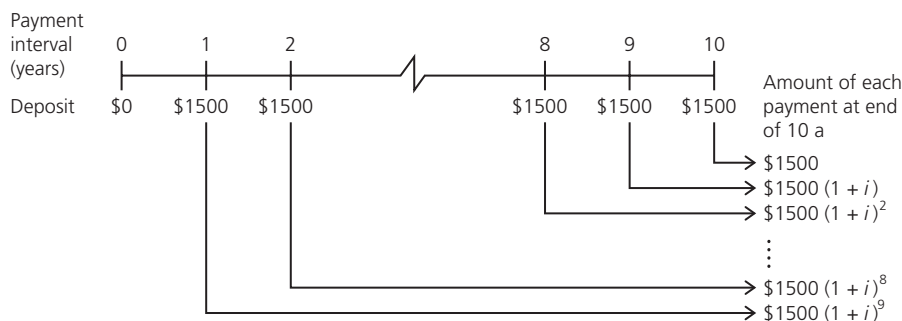
Example 3

Leno deposits \$1500 at the end of each year for ten years in a registered home ownership plan (RHOSP) that earns 10%/a, compounded quarterly.

- Draw a time line to represent the annuity.
- Write the series that represents the annuity.
- How much money will have accumulated in the plan at the end of ten years?
- Verify your answer using technology.
- What would be the accumulated value if the plan earned 10%/a, compounded annually? Comment on the difference.

Solution

- Draw a time line that shows the payment intervals, not the interest conversion periods. Therefore, $n = 10$.



- The payments are annual, but the interest is compounded quarterly. Before you can write the sequence, you need to find the nominal interest rate that is equivalent to 10%/a, compounded quarterly. Let i be 2.5%. Then

$$1 + i = 1.025^4$$

$$i = 1.025^4 - 1$$

$$i \doteq 0.103\,812\,891$$

The geometric series is $S_{10} = 1500 + 1500(1.103\,812\,891) + 1500(1.103\,812\,891)^2 + \dots + 1500(1.103\,812\,891)^8 + 1500(1.103\,812\,891)^9$.

- (c) Find the sum of the series using $S_n = \frac{a(r^n - 1)}{r - 1}$, where $a = 1500$, $r = 1.103\,812\,891$, and $n = 10$. Or, find the accumulated value of the ordinary simple annuity using $FV = R \times \frac{(1 + i)^n - 1}{i}$, where $R = 1500$, $n = 10$, and $i = 0.103\,812\,891$.

$$S_{10} = FV = 1500 \times \frac{(1.103\,812\,891)^{10} - 1}{0.103\,812\,891}$$

Note: It is important to use as many digits as possible when calculating. Avoid rounding numbers until your final calculation.

$$FV = \$24\,347.61$$

At the end of ten years, \$24 347.61 will have accumulated in Leno's plan. This amount represents a good down payment for the purchase of a home.

- (d) Use the **TVM Solver** of the TI-83 Plus calculator to verify the solution. Do not find the equivalent nominal rate. You can enter the information that you have directly into the **TVM Solver**, and you will see why in the next few steps.

Press **[MODE]** and change the fixed decimal mode to 2, because most of the values in this section represent dollars and cents.

Open the **TVM Solver**. Enter the following:

$N = 10$, since there are ten payment intervals

$I\% = 10$, the annual interest rate

$PV = 0$

$PMT = -1500$. This value is negative, because the payment is an outflow of Leno's cash.

$FV = 0$

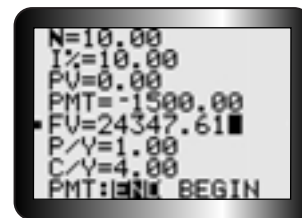
$P/Y = 1$, since only one deposit is made annually

$C/Y = 4$, since interest is compounded quarterly

Scroll to the line containing **FV** and press **[ALPHA]** **[ENTER]**.

The future or accumulated value is \$24 347.61, as it is in (a).

Notice that the value is positive, because it represents an inflow of cash for Leno.



- (e) Find the accumulated value of the ordinary simple annuity with $R = 1500$, $n = 10$, and $i = 0.10$.

$$\begin{aligned} FV &= R \times \frac{(1 + i)^n - 1}{i} && \text{Substitute the values for } R, n, \text{ and } i. \\ &= 1500 \times \frac{(1.10)^{10} - 1}{0.10} && \text{Simplify.} \\ &= 23\,906.14 \end{aligned}$$

The accumulated value is \$23 906.14, if the plan earned interest at 10%/a, compounded annually. This value is \$441.47 less than the value earned if the interest is compounded quarterly over ten years.

The length or frequency of the interest conversion period has some effect on the accumulated value. But this effect seems small when compared to the effects that the interest rate and the size of payment have on the accumulated value of an annuity.

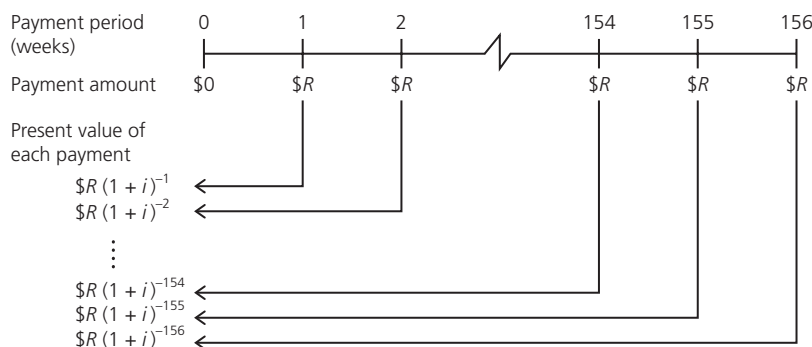
Example 4

Jade Coyne wishes to purchase a used car from her uncle. She will pay him \$18 000 by making weekly payments. They agree that Jade will pay interest at 7.2%/a, compounded monthly. Jade would like to repay the loan in three years.

- Draw a time line to represent the repayment of the loan.
- Write the series that represents the repayment of the loan.
- What is the weekly payment?
- Use technology to determine how much sooner Jade could repay the loan if the weekly payment is increased by \$20.

Solution

- Draw a time line that shows the payment intervals, not the interest conversion periods. Therefore, $n = 3 \times 52$ or 156.



- Find the interest rate, i , compounded weekly, that is equivalent to 7.2%/a, compounded monthly. $7.2\% \div 12 = 0.6\%$

$$(1 + i)^{52} = (1.006)^{12} \quad \text{Take the 52nd root of each side.}$$

$$1 + i = \sqrt[52]{(1.006)^{12}}$$

$$1 + i = (1.006)^{\frac{12}{52}}$$

$$i = (1.006)^{\frac{3}{13}} - 1$$

$$i \doteq 0.001\,381\,431\,4$$

The series is $S_{156} = R(1.001\,381\,431\,4)^{-156} + R(1.001\,381\,431\,4)^{-155} + R(1.001\,381\,431\,4)^{-154} + \dots + R(1.001\,381\,431\,4)^{-2} + R(1.001\,381\,431\,4)^{-1}$, where $S_{156} = 18\,000$.

(c) Finding the Weekly Payment Using Formulas

You could find the sum of the geometric series using $S_n = \frac{a(r^n - 1)}{r - 1}$, with $a = R(1.001\,381\,431\,4)^{-156}$, $n = 156$, $S_{156} = 18\,000$, and $r = 1.001\,381\,431\,4$. Then solve for R .

Alternatively, you can find the weekly deposit, R , using the formula for the present value of an ordinary simple annuity.

$PV = 18\,000$, $n = 156$, and $i = 0.001\,381\,431\,4$

$$PV = R \times \frac{1 - (1 + i)^{-n}}{i} \quad \text{Substitute the values for } PV, n, \text{ and } i.$$

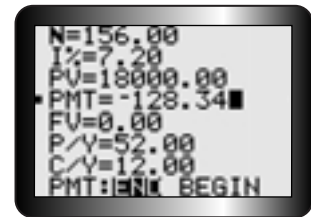
$$18\,000 = R \times \frac{1 - (1.001\,381\,431\,4)^{-156}}{0.001\,381\,431\,4} \quad \text{Simplify and solve for } R.$$

$$R = \$128.34$$

Finding the Weekly Payment Using Technology

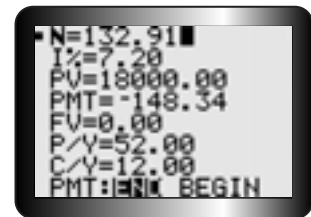
Solve for the payment, R , using the TVM Solver. Enter the values as shown in the screen, except for PMT. Scroll to the line containing PMT and press **[ALPHA]** **[ENTER]**.

The payment is \$128.34. The value is negative in the screen because the payment is an outflow of cash for Jade.



- (d) Open the TVM Solver and adjust the payment to $PMT = -148.34$. Scroll to the line containing N and press **[ALPHA]** **[ENTER]**.

Jade will repay the loan after about 133 weeks, or 23 weeks sooner, if she increases the monthly payment by \$20.



Consolidate Your Understanding

1. An investment earns 12%/a, compounded semiannually. Would the equivalent nominal rate, compounded quarterly, be greater than or less than 12%/a? Explain why.
2. Write a problem so that the answer to the problem is $(1 + i)^2 = \left(1 + \frac{0.09}{4}\right)^4$.
3. Solve for i in question 2.

Key Ideas

- An annuity where the payment interval does not correspond to the interest conversion period is called a **general annuity**.
- Two interest rates are said to be **equivalent** if they yield the same accumulated values at the end of one year.
- The **nominal** rate of interest is the annual rate.
- Solve an ordinary general annuity by first finding the nominal interest rate that coincides with the payment interval. Then solve the problem as you would for an ordinary simple annuity.
- Use the time-value-of-money (TVM) solver of the TI-83 Plus calculator to solve simple and general annuity problems. The **TVM Solver** allows you to specify both the **P/Y**, the payments per year, and **C/Y**, the number of interest conversion periods per year. For this reason, it is not necessary to find the equivalent rate when solving a general annuity.

Practise, Apply, Solve 2.10

A

1. Simplify.

(a) $\sqrt{(1+i)^2}$ (b) $\sqrt[4]{(1+i)^4}$ (c) $\sqrt[7]{(1+i)^7}$ (d) $\sqrt[3]{(1+i)^3}$

2. Evaluate to four decimal places.

(a) $\sqrt{(1.06)^4}$ (b) $\sqrt[3]{(1.08)^{12}}$ (c) $\sqrt[4]{(1.0125)^{12}}$ (d) $\sqrt{(1.04)^{18}}$

3. Solve for i .

(a) $1 + i = (1.025)^4$ (b) $1 + i = (1.01)^{12}$ (c) $(1 + i)^2 = 1.10$
 (d) $(1 + i)^{12} = 1.015$ (e) $(1 + i)^4 = (1.06)^2$ (f) $(1 + i)^{12} = (1.025)^4$
 (g) $(1 + i)^{52} = (1.03)^4$ (h) $(1 + i)^2 = (1.0055)^{52}$

4. Find the equivalent nominal interest rate, compounded annually and rounded to two decimal places, for each of the following.

(a) 8%/a, compounded semiannually (b) 8%/a, compounded quarterly
 (c) 10.4%/a, compounded weekly (d) 8.76%/a, compounded daily
 (e) 18%/a, compounded quarterly (f) 21%/a, compounded monthly

5. Find the nominal rate that is equivalent to 18%/a, compounded quarterly, if interest is paid

(a) annually (b) semiannually (c) monthly
 (d) weekly (e) daily

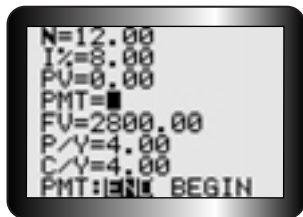
B

6. For each of the following,
- draw a time line to represent the amount of each annuity
 - write the series that represents the amount of the annuity
 - find the amount of each on the date of the last payment
- (a) \$5000 is deposited at the end of every year for 10 years at 8%/a, compounded semiannually
- (b) \$800 is deposited at the end of every 3 months for 5 years at 8%/a, compounded annually
- (c) \$150 is deposited at the end of every week for 2 years at 11%/a, compounded semiannually
- (d) \$3300 is deposited at the end of every 6 months for 7 years at 10%/a, compounded quarterly
- (e) \$1000 is deposited at the end of every month for 3 years at 9%/a, compounded annually
7. Michelle wants to purchase a new stereo sound system for \$3800. She can put \$300 down and finance the outstanding balance. The interest rate is 18%/a, compounded annually. What monthly payment will allow her to pay for the purchase in 18 months? Include a time line.
8. **Knowledge and Understanding:** Natalie and Preston would like to accumulate \$25 000 at the end of 3.5 years for a future down payment on a house in Prescott. How much should they deposit at the end of each week in a savings account that pays 7.2%/a, compounded monthly, to meet their goal?
- Draw a time line to represent the annuity.
 - Write the series that represents the amount of the annuity.
 - Determine the weekly deposit.
9. Recall question 9 in section 2.8. Nader plans to buy a used car. He can afford to pay \$280 at the end of each month for three years. The best interest rate he can find is 9.8%/a, compounded monthly. For this interest rate, the most he could spend on a vehicle is \$8702.85.
- Determine the amount he could spend on the purchase of a car if the interest rate is 9.8%/a, compounded annually.
 - Explain why the change in the compounding period has such a small effect on the amount he can afford to spend on a car.

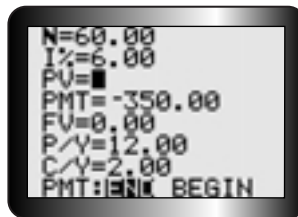


- 10. Communication:** The following screens were obtained from the TVM Solver of the TI-83 Plus calculator. Write a problem that would describe each screen.

(a)



(b)



- 11. Application:** Daniel buys an engagement ring for his fiancée, Lise. He pays \$3400 by putting \$500 down and financing the rest at 9.6%/a, compounded annually. He hopes to completely pay this debt in three years. What must he pay at the end of each month? Verify your answer using technology.

If possible, use technology to solve questions 12 to 18. Use the TVM Solver on the TI-83 Plus calculator. Or visit the Nelson Thomson Learning Web site at www.math.nelson.com and use one of the several online loan calculators.

Jack and Jill Hill are saving money to purchase their first home. Their intent is to deposit \$832 at the end of each month for six years in a savings account that pays 7.2%/a, compounded annually. They have hired you to analyze the effects of changing the conditions of their long-term savings plan to maximize the money they can save. Questions 12 to 16 will assist you in your analysis.

- 12.** First analyze the effect of changing the compounding period.
- Determine the future value of their savings plan at the end of six years, if the savings account pays
 - 7.2%/a, compounded annually
 - 7.2%/a, compounded semiannually
 - 7.2%/a, compounded monthly
 - 7.2%/a, compounded weekly
 - 7.2%/a, compounded daily
 - Based on your analysis, describe the effect of changing the compounding period on their savings plan.
- 13.** Analyze the effect of altering the frequency of the deposits.
- The Hills deposit \$832 at the end of each month. What is the equivalent year-end deposit? Use your answer to find the future value of the annual deposits after six years.
 - What is the equivalent semiannual deposit? Find the future value of their savings plan after six years using the semiannual deposit.
 - Determine the equivalent weekly deposit. Use it to find the future value of their savings plan after six years.
 - Determine the equivalent daily deposit. Use it to find the future value of their savings plan after six years.
 - Based on your analysis, report the effect that the frequency of the deposits has on their savings plan.

14. Analyze the effect of changing the interest rate.
 - (a) Determine the future value of their savings plan at the end of six years if the interest earned on the monthly deposits of \$832 is
 - i. 7.8%/a, compounded annually
 - ii. 8.4%/a, compounded annually
 - iii. 6%/a, compounded annually
 - (b) Based on your analysis, report the effect of changing the interest rate on their savings plan.
15. Analyze the effect of changing the size of the deposit.
 - (a) Determine the future value of their savings plan at the end of six years if the monthly deposits of \$832 are
 - i. increased by 10%
 - ii. decreased by 10%
 - iii. changed to \$800
 - (b) Based on your analysis, report the effect of changing the size of the deposit on their savings plan.
16. Analyze the effect of the total number and timing of the deposits.
 - (a) Determine the future value of their savings plan at the end of six years if the monthly deposits of \$832 each are made at the beginning of each month, instead of at the end of each month.
 - (b) Determine the future value of their savings plan at the end of six years if the Hills wait one year before deciding to make their month-end deposits of \$832 each.
 - (c) Determine the future value of their savings plan at the end of six years if the Hills decide to stop making deposits at the end of the fifth year.
 - (d) Determine the future value of their savings plan if the Hills decide to make deposits for 6.5 years by starting the plan six months earlier.
 - (e) Based on your analysis, report on the effect of changing the total number and timing of the deposits on their savings plan.
17. Based on what you have analyzed, create a report that you could present to the Hills. In your report, outline the best and most important strategies they should follow to save as much money as possible for their home purchase.
18. An ad for a big-screen television reads, “Was \$3270. Now \$2890 cash, or \$500 down and \$122.81 for 24 months on approved credit.” Use the financial functions of the TI-83 Plus calculator to answer the following.
 - (a) What is the annual rate of interest, compounded monthly, of the payment plan?
 - (b) What is the total interest?
19. **Check Your Understanding:** Explain why 12%/a, compounded monthly, is not the same as 12%/a, compounded quarterly.

C

20. Lee wishes to purchase a downhill ski package that includes the skis, poles, bindings, and boots. The selling price is \$1894. The finance plan includes a \$150 down payment and payments of \$113 each at the end of the month for 1.5 years. Find the annual interest charge, if the interest charge is
- (a) compounded monthly (b) compounded annually
21. **Thinking, Inquiry, Problem Solving**
- (a) Imagine that you have a loan and the interest rate on the loan doubles. You wish to keep the same amortization period, but should you double the payment? Justify your reasoning with examples.
- (b) Imagine that you are going to borrow money. You decide to double the amount that you want to borrow. Should you double the payment if you wish to keep the same amortization period? Justify your reasoning with examples.



The Chapter Problem—Financial Planning

In this section, you studied general annuities. Apply what you learned to solve these questions about the Chapter Problem on page 106.

- CP16.** What interest rate, compounded monthly, is equivalent to $8.4\%/a$, compounded semiannually? Without technology, use this rate to find Mr. Sacchetto's new monthly mortgage payments. Recall that he owes \$77 000 and has 20 years remaining in the term. Determine the principal still owing on his mortgage after five years.
- CP17.** Use the **TVM Solver** to verify the new payment and outstanding balance.
- CP18.** Recall that, at age 55, Mr. Sacchetto may choose between two RIF options for investing the \$120 000 savings. Option A provides monthly payments from an RIF that pays $7.2\%/a$, compounded monthly. Option B provides monthly payments from an RIF that pays $7.25\%/a$, compounded quarterly. Without using the **TVM Solver**, determine the monthly RIF payment Mr. Sacchetto would receive if he selects Option B.
- CP19.** Write the values that you would enter for **N**, **I%**, **PV**, **FV**, **P/Y**, and **C/Y** into the **TVM Solver** to find the monthly RIF payment Mr. Sacchetto would receive if the fund pays $7.25\%/a$, compounded quarterly. Verify your solution for question CP18.
- CP20.** Which RIF payment option should he choose?