

Functions

Using Properties of Relations to Sketch Their Graphs

If the algebraic expression of a relation can be identified as linear or quadratic, its graph can be sketched without making a table or using graphing technology.

Linear Relations

The general form of a linear relation is $ax + by + c = 0$. Once the relation is identified as linear, it can be graphed using **i.** the x - and y -intercepts or **ii.** the slope, m , and y -intercept b , when $y = mx + b = 0$.

Example 1

Sketch the graph of $3x + 5y + 15 = 0$.

Solution

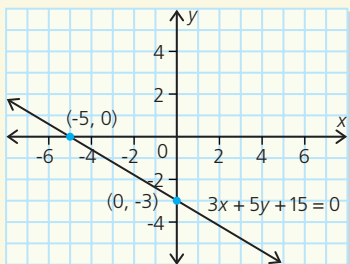
The graph is linear. Use the x - and y -intercepts to draw the graph.

Determine the x -intercept by letting $y = 0$.

$$\begin{aligned} 3x + 5(0) + 15 &= 0 \\ 3x &= -15 \\ x &= -5 \end{aligned}$$

Determine the y -intercept by letting $x = 0$.

$$\begin{aligned} 3(0) + 5y + 15 &= 0 \\ 5y &= -15 \\ y &= -3 \end{aligned}$$



Example 2

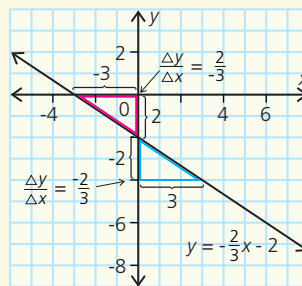
Sketch the graph of $3y = -2x - 6$.

Solution

Rewrite the relation in the form $y = mx + b$.

Then, $y = -\frac{2}{3}x - 2$. Mark the y -intercept as -2 and use the slope to locate a second point.

The slope is $m = -\frac{2}{3}$. Then $\frac{\Delta y}{\Delta x} = \frac{-2}{3}$ or $\frac{2}{-3}$.



Quadratic Relations in Standard Form

The standard form of a quadratic relation is $y = ax^2 + bx + c$. The graph is a parabola. If $a > 0$, the parabola opens upward; if $a < 0$, the parabola opens downward.

Example 3: Graphing Using Symmetry

Sketch the graph of $y = -3x^2 - 2x + 7$.

Solution

$$y = -3x^2 - 2x + 7 \quad a = -3, \text{ so the parabola opens downward.}$$

$$y = x(-3x - 2) + 7 \quad \text{Factor partially.}$$

Let $x = 0$ or $-3x - 2 = 0$ to find two points on the curve. When $x = 0$, then $y = 7$.

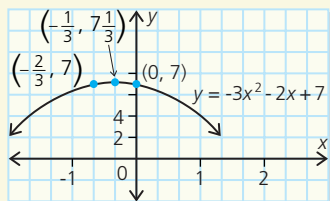
When $-3x - 2 = 0$, then $x = -\frac{2}{3}$, and $y = 7$. The axis of symmetry is halfway between $(0, 7)$ and $(-\frac{2}{3}, 7)$.

$$\text{Therefore, } x = \frac{0 + (-\frac{2}{3})}{2} = -\frac{1}{3}$$

To find the vertex, substitute $x = -\frac{1}{3}$ into $y = -3x^2 - 2x + 7$.

$$y = -3\left(-\frac{1}{3}\right)^2 - 2\left(-\frac{1}{3}\right) + 7 = 7\frac{1}{3}$$

The curve opens downward, so the vertex occurs at $(-\frac{1}{3}, 7\frac{1}{3})$.



Quadratic Relations in Vertex Form

The vertex form of a quadratic relation is $y = a(x - h)^2 + k$, where (h, k) is the vertex. This is also the coordinates of the maximum point when $a < 0$ and of the minimum point when $a > 0$.

Example 4: Graphing Using Vertex Form

Sketch the graph of $y = 2(x + 2)^2 + 3$.

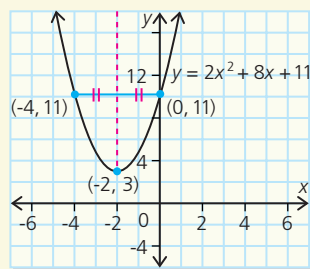
Solution

The equation is in vertex form, and we see that the vertex is $(-2, 3)$. Determine one point on the curve and use symmetry to find a second point.

When $x = 0$,

$$y = 2(0 + 2)^2 + 3 = 11$$

So, $(0, 11)$ is a point on the curve. Another point, $(-4, 11)$, is symmetric to the axis of symmetry. Now sketch.



In this case, $(-2, 3)$ is the minimum point. The relation has a minimum value of 3, when $x = -2$.

Practice

1. Sketch each graph using intercepts.
 - (a) $4x - 6y = 12$
 - (b) $-5x + 3y = 15$
 - (c) $3x - 8y + 24 = 0$
 - (d) $\frac{1}{2}x - \frac{3}{4}y = 9$
 2. Sketch the graphs using the slope and y -intercept.
 - (a) $y = -3x + 7$
 - (b) $y = -\frac{3}{4}x - 2$
 - (c) $2x + 3y = 6$
 - (d) $\frac{2}{3}y = 6x + 1$
 3. Sketch the graphs using partial factoring.
 - (a) $y = 2x^2 - 6x + 5$
 - (b) $y = -3x^2 + 9x - 2$
 - (c) $y = 5x^2 - 3 + 5x$
 - (d) $y = 3 + 4x - 2x^2$
 4. Sketch the graphs using the zeros of the curve.
 - (a) $y = x^2 + 4x - 12$
 - (b) $y = x^2 - 7x + 10$
 - (c) $y = 2x^2 - 5x - 3$
 - (d) $y = 6x^2 - 13x - 5$
 5. Sketch the graph.
 - (a) $y = (x - 2)^2 + 3$
 - (b) $y = (x + 4)^2 - 10$
 - (c) $y = 2(x - 1)^2 + 3$
 - (d) $y = -3(x + 1)^2 - 4$
 6. A football is kicked into the air. Its height, h , in metres, after t seconds is approximated by $h = -5(t - 2.3)^2 + 27.45$. What is the maximum height reached by the football?
 7. A computer game company models the profit on its latest game using $P = -3\left(x - \frac{19}{3}\right)^2 - \frac{525}{12}$, where x is the number of games sold in hundred thousands, and P is the profit in millions of dollars. What is the maximum profit the company can expect to earn using this model?
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Completing the Square to Convert to the Vertex Form of a Parabola

A quadratic relation in standard form, $y = ax^2 + bx + c$, can be rewritten in vertex form as $y = a(x - h)^2 + k$ by creating a perfect square in the original and then factoring the square.

Steps to Complete the Square	Example: $y = 2x^2 + 12x - 5$
• Remove the common constant factor from both the x^2 - and x -term.	$y = 2(x^2 + 6x) - 5$
• Determine the constant that must be added (and subtracted) to create a perfect square. This is half the coefficient of the x -term, squared.	$y = 2(x^2 + 6x + 9 - 9) - 5$
• Group the three terms of the perfect square. Multiply the subtracted value and move it outside the bracket.	$y = 2(x^2 + 6x + 9) - 2(9) - 5$
• Factor the perfect square and collect like terms.	$y = 2(x + 3)^2 - 23$

Practice

- Write each trinomial as a perfect square.
 - $x^2 + 2x + 1$
 - $x^2 + 4x + 4$
 - $x^2 + 6x + 9$
 - $x^2 + 10x + 25$
 - $x^2 - 12x + 36$
 - $x^2 - 14x + 49$
 - $x^2 - 18x + 81$
 - $x^2 - 20x + 100$
- Complete the square, and write in vertex form.
 - $y = x^2 + 2x + 2$
 - $y = x^2 + 4x + 6$
 - $y = x^2 - 12x + 40$
 - $y = x^2 - 18x + 80$
- Express in vertex form by completing the square. State the equation of the axis of symmetry and the coordinates of the vertex.
 - $y = 2x^2 - 4x + 7$
 - $y = 5x^2 + 10x + 6$
 - $y = -3x^2 - 12x + 2$
 - $y = -\frac{1}{2}x^2 + 5x + 2$
 - $y = -0.1x^2 - 0.5x + 0.4$
 - $y = -\frac{3}{4}x^2 + 5x - 2$
- Complete the square to find the roots.
 - $x^2 - 10x + 9 = 0$
 - $9x^2 - 12x - 5 = 0$
 - $2x^2 + 3x - 20 = 0$
- A baseball is hit from a height of 1 m. Its height in metres, h , after t seconds is $h = -4.9t^2 + 10t + 1$.
 - What is the maximum height of the ball?
 - When does the ball reach this height?
- A town's population is $P = 0.16t^2 + 7.2t + 100$, where P is the population in thousands and t is the time in years, with $t = 0$ being the year 2000. When will the population reach its minimum? What is the minimum?

Solving Quadratic Equations: The Quadratic Formula

The solutions, or roots, of a quadratic equation in the form $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 1

Solve $3x^2 - 2x = 6$ to two decimal places.

Solution

Rewrite $3x^2 - 2x = 6$ in the form $ax^2 + bx + c = 0$. Then $3x^2 - 2x - 6 = 0$, where $a = 3$, $b = -2$, and $c = -6$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute known values.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{76}}{6}$$

Then, $x = 1.79$ and $x = -1.12$ to two decimal places.

The discriminant, $b^2 - 4ac$, can be used to determine the nature of the roots.

- If $b^2 - 4ac > 0$, there are two real roots.
- If $b^2 - 4ac = 0$, there is one real root.
- If $b^2 - 4ac < 0$, there are two imaginary, or non-real, roots.

Example 2

Determine the number and type of roots of $5m^2 = 3m - 7$.

Solution

Rewrite $5m^2 = 3m - 7$ in the form $ax^2 + bx + c = 0$. Then $5m^2 - 3m + 7 = 0$, where $a = 5$, $b = -3$, and $c = 7$. Determine the number and type of roots by examining $b^2 - 4ac$.

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4(5)(7) \\ &= -131 \end{aligned}$$

Then $b^2 - 4ac < 0$, so there are two imaginary or non-real roots.

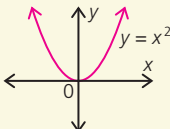
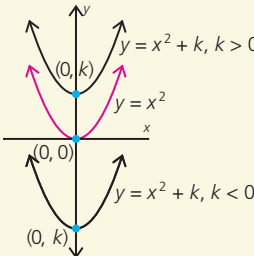
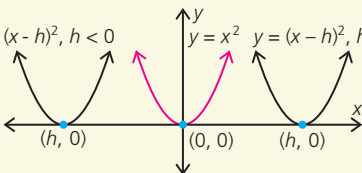
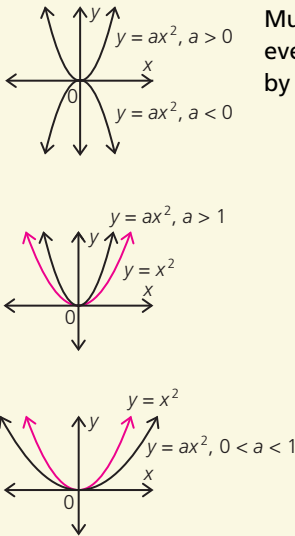
Practice

Evaluate all answers to two decimal places where necessary.

1. Indicate whether each equation has two real roots, one real root, or two imaginary or non-real roots.
 - (a) $3m^2 - 2m + 9 = 0$
 - (b) $15x^2 - 6 = x$
 - (c) $9m^2 - 24m = -16$
 - (d) $36 + 25n^2 = -60n$
 - (e) $20x^2 + 9x - 18 = 0$
 - (f) $5x^2 = 6x - 7$
 2. Solve each equation using the quadratic formula.
 - (a) $10x^2 - x - 7 = 0$
 - (b) $15m^2 + 27m + 20 = 0$
 - (c) $28p^2 - 71p = -45$
 - (d) $42y^2 + y = 30$
 - (e) $32 - 76x + 45x^2 = 0$
 - (f) $35 + 9x - 18x^2 = 0$
 3. A water balloon is catapulted into the air so that its height, h , in metres, after t seconds is $h = -4.9t^2 + 25t + 1.9$.
 - (a) When is the balloon 13 m high?
 - (b) How long does the balloon take to hit the ground?
 4. The underside of a bridge is a parabolic arch defined by $h = -0.1d^2 + 2d$, where h is the height in metres above the bottom of the bridge and d is the distance in metres from the left side of the bridge. How far from the left side of the bridge is the height 5 m? Give the answer to one decimal.
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Transformations of Quadratic Relations

The graph of any quadratic relation can be created by altering or repositioning the graph of the base curve $y = x^2$. To do so, write the relation in vertex form, $y = a(x - h)^2 + k$. The base curve, $y = x^2$, is translated vertically or horizontally, stretched or compressed vertically, or reflected about the x -axis, depending on the values of a , h , and k .

Relation	Type of Transformation	Graph	Explanation
$y = x^2$	Base parabola		This is the base curve upon which other transformations are applied.
$y = x^2 + k$	Vertical Translation The curve shifts up if $k > 0$ and down if $k < 0$.		Add k to the y -coordinate of every point on the base curve. The resultant curve is congruent to the base curve.
$y = (x - h)^2$	Horizontal Translation The curve shifts to the right if $h > 0$ and to the left if $h < 0$.		Subtract h from the x -coordinate of every point on the base curve.
$y = ax^2$	Reflection The curve is reflected about the x -axis if $a < 0$. Vertical Stretch The curve has a narrow opening if $a > 1$. Vertical Compression The curve has a wide opening if $0 < a < 1$.		Multiply the y -coordinate of every point on the base curve by a .

Example

Use transformations to graph $y = -2(x + 3)^2 - 4$.

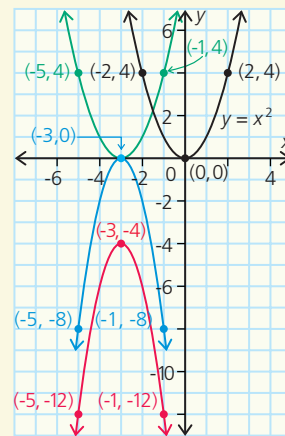
Solution

Step 1: Begin with the graph of the base curve, $y = x^2$ (black). Select three points on the curve to help define its shape. From the form $y = a(x - h)^2 + k$, see that $a = -2$, $h = -3$, and $k = -4$.

Step 2: $h = -3$, so shift the entire parabola 3 units to the left (green).

Step 3: $a = -2$, so multiply every y -coordinate by -2 . The parabola is reflected and its opening becomes narrower (blue).

Step 4: $k = -4$, so shift the entire parabola 4 units down. This is the final graph (red).



Practice

In questions 1 and 2, the coordinates of a point on the parabola $y = x^2$ are given. State the new coordinates under each transformation.

- (2, 4); shift up 3
 - (-2, 4); shift down 5
 - (-1, 1); shift left 4
 - (1, 1); shift right 6
 - (3, 9); vertical stretch of $-\frac{1}{3}$
 - (2, 4); vertical stretch of 2
- (-2, 4); shift left 2 and up 5
 - (-1, 1); shift right 4, vertical stretch 3, and reflect about x -axis
 - (3, 9); vertical compression by 3, shift left 5, and shift down 2
 - (0, 0); shift left 5, vertical stretch 3, shift up 4, and reflect about x -axis
- Points (-2, 4), (0, 0), and (2, 4) are on the parabola $y = x^2$. Use your knowledge

of transformations to determine the equation of the parabola using these coordinates.

- (-2, 6), (0, 2), (2, 6)
- (-2, 12), (0, 0), (2, 12)
- (2, 4), (4, 0), (6, 4)
- (-2, -6), (0, -2), (2, -6)

In questions 4 and 5, sketch each graph using transformations on $y = x^2$.

- $y = x^2 + 3$
 - $y = x^2 - 4$
 - $y = (x - 3)^2$
 - $y = (x + 4)^2$
 - $y = 2x^2$
 - $y = \frac{1}{2}x^2$
- $y = 2(x - 1)^2 + 3$
 - $y = -3(x + 1)^2 + 2$
 - $y = \frac{1}{2}(x - 2)^2 - 1$
 - $y = -\frac{1}{3}(x + 2)^2 - 4$