

In mathematics, you must be able to represent intervals and identify smaller sections of a relation or a set of numbers. You can use the following inequality symbols:

$>$ greater than

$<$ less than

$\geq$ greater than or equal to

$\leq$ less than or equal to

When you write one of these symbols between two or more expressions, the result is called an **inequality**, for example, $3x - 1 < 8$.

To solve an inequality you have to find all the possible values of the variable that satisfy the inequality.

For example, $x = 2$ satisfies $3x - 1 < 8$, but so does $x = 2.9$, $x = -1$, and $x = -5$.



In fact, every real number less than 3 satisfies this inequality. The thicker part of the number line below represents this solution. To show that 3 is not included in the solution, use an open dot at this value. A solid dot shows that a value is included in the solution.

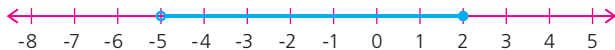


The solution to $3x - 1 < 8$ can be written in set notation as $\{x \mid x < 3, x \in \mathbf{R}\}$. This is read “the set of x such that x is less than 3, where x is an element of the set of real numbers.”

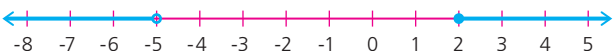
Example 1

Write each set in set notation.

(a)



(b)



Solution

(a) $\{x \mid -5 < x \leq 2, x \in \mathbf{R}\}$

(b) $\{x \mid x < -5 \text{ or } x \geq 2, x \in \mathbf{R}\}$

Example 2

Graph each set on a number line.

(a) $\{x \mid -5 < x \leq 2, x \in \mathbf{N}\}$

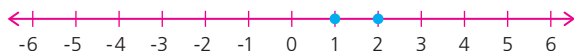
(b) $\{x \mid -1 < x \leq 1, x \in \mathbf{R}\}$

(c) $\{x \mid x^2 \leq 9, x \in \mathbf{R}\}$

(d) $\{x \mid x^2 > 4, x \in \mathbf{R}\}$

Solution

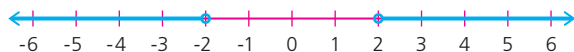
(a) Recall that $\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}$.



(c) For $x^2 \leq 9$, $x \leq 3$ and $x \geq -3$.



(d) For $x^2 > 4$, $x > 2$ or $x < -2$.



In earlier grades, you learned to solve linear equations algebraically

- by adding or subtracting the same quantity from both sides, or
- by multiplying or dividing both sides by a nonzero quantity

To solve a linear inequality such as $3x - 1 < 8$ algebraically, the rules are similar, but not identical.

Start with the inequality $3 < 4$.

Add 7 to both sides, then $10 < 11$. The left side is still less than the right side.

Add -7 to both sides, then $-4 < -3$. The left side is still less than the right side.

Multiply both sides by 2, then $6 < 8$. The left side is still less than the right side.

Multiply both sides by -2 , then $-6 > -8$. The left side is **greater** than the right side now.

Divide both sides by 2, then $1.5 < 2$. The left side is still less than the right side.

Divide both sides by -2 , then $-1.5 > -2$. The left side is **greater** than the right side now.

The same rules apply to an inequality that has more than two expressions, for example, $-2 \leq 3 < 4$.

Add or subtract the same number from all three parts of this inequality. Notice that the inequality signs do not change.

Multiply or divide all three parts of this inequality by the same number. Notice that the inequality signs change if each part is multiplied or divided by a negative number.

Example 3

Solve and graph each solution set on a number line, $x \in \mathbf{R}$.

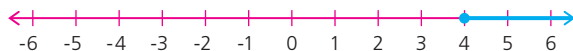
(a) $1 - x \leq -3$ (b) $\frac{2a}{3} + 4 > 2$ (c) $-4 < \frac{1 - 3x}{2} < 1$

Solution

(a) $1 - x \leq -3$ Add x to both sides.
 $1 \leq -3 + x$ Add 3 to both sides.
 $4 \leq x$

This is equivalent to $x \geq 4$.

The solution set is $\{x \mid x \geq 4, x \in \mathbf{R}\}$.



(b) $\frac{2a}{3} + 4 > 2$ Subtract 4 from both sides.
 $\frac{2a}{3} > -2$ Multiply both sides by 3.
 $2a > -6$ Divide both sides by 2.
 $a > -3$

The solution set is $\{a \mid a > -3, a \in \mathbf{R}\}$.



(c) $-4 < \frac{1 - 3x}{2} < 1$ Multiply each expression by 2.
 $-8 < 1 - 3x < 2$ Subtract 1 from each expression.
 $-9 < -3x < 1$ Divide each expression by -3 and reverse the inequality signs.
 $3 > x > -\frac{1}{3}$ or $-\frac{1}{3} < x < 3$

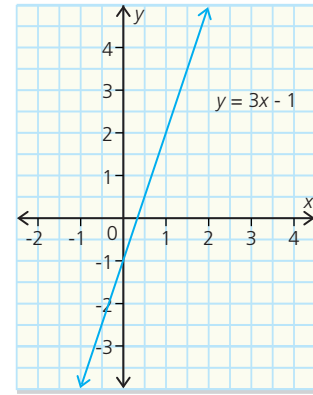
The solution set is $\{x \mid -\frac{1}{3} < x < 3, x \in \mathbf{R}\}$.



Example 4

The graph of $f(x) = 3x - 1$ is shown.

- (a) What can you say about $f(x)$ if $x > 2$?
 (b) For $2 < f(x) < 8$, what is the restriction on x ?



Solution

- (a) In the graph of $y = f(x)$, the green area represents $x > 2$. Follow the arrow up and then over to the y -axis. The yellow area represents $y > 5$ or $f(x) > 5$.

You can also find the **restriction** on y , where $y = 3x - 1$, algebraically. Start with $x > 2$.

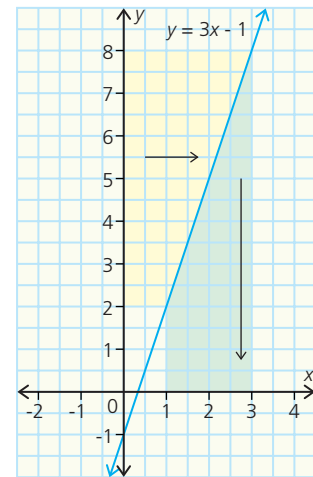
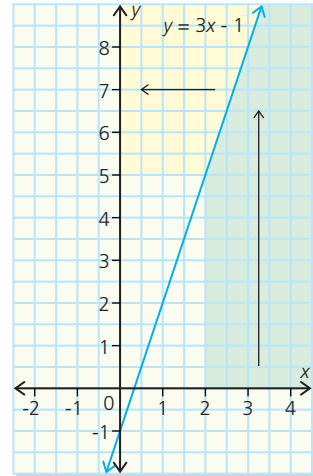
$$\begin{array}{ll} x > 2 & \text{Multiply both sides by 3.} \\ 3x > 6 & \text{Subtract 1 from both sides.} \\ 3x - 1 > 5 & \end{array}$$

Then $f(x) > 5$.

- (b) For $2 < f(x) < 8$, the y -coordinates must be between 2 and 8, which is represented in the graph by the yellow area. Follow the arrow right and then down to the x -axis. The green area shows that x must be between 1 and 3.

You can also find the restriction on x algebraically.

$$\begin{array}{ll} \text{If } 2 < f(x) < 8, & \\ \text{then, } 2 < 3x - 1 < 8 & \text{Add 1 to each expression.} \\ 3 < 3x < 9 & \text{Divide each expression by 3.} \\ 1 < x < 3 & \end{array}$$



Example 5

Find the range of the function if $-2 < x \leq 7$ and $f(x) = 3x + 1$.

Solution

$$\begin{array}{ll} -2 < x \leq 7 & \text{Multiply each expression by 3.} \\ -6 < 3x \leq 21 & \text{Add 1 to each expression.} \\ -5 < 3x + 1 \leq 22 & \text{Replace } 3x + 1 \text{ with } f(x). \\ -5 < f(x) \leq 22 & \end{array}$$

The range of the function is the set of all real numbers greater than -5 but less than or equal to 22 . So the range is $\{y \mid -5 < y \leq 22, y \in \mathbf{R}\}$.

Key Ideas

- To solve an inequality means to find all the possible values of the variable that satisfy the inequality.
- To solve an inequality algebraically, isolate the variable by
 - ♦ adding or subtracting the same quantity from both sides
 - ♦ multiplying or dividing both sides by a positive quantity
 - ♦ multiplying or dividing both sides by a negative quantity and reverse the inequality signs

Practise, Apply, Solve 3.3

A

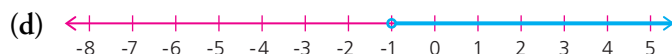
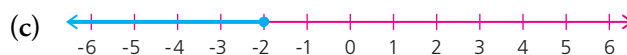
1. Graph each of the following on a number line.

“Or” means that if either test is satisfied, then the value belongs to the set.

“And” requires that both tests are satisfied before the value is included in the set.

- | | |
|---|--|
| (a) $\{x \mid 3 < x \leq 5, x \in \mathbf{N}\}$ | (b) $\{x \mid x \geq -1, x \in \mathbf{R}\}$ |
| (c) $\{s \mid s < -3, s \in \mathbf{R}\}$ | (d) $\{t \mid 1 < t < 2, t \in \mathbf{R}\}$ |
| (e) $\{x \mid -3 \leq x < 1, x \in \mathbf{R}\}$ | (f) $\{u \mid 3 \geq u \geq 1, u \in \mathbf{R}\}$ |
| (g) $\{x \mid x < -3 \text{ or } x > 2, x \in \mathbf{R}\}$ | (h) $\{x \mid x < 5 \text{ and } x \geq 2, x \in \mathbf{R}\}$ |
| (i) $\{x \mid x^2 \leq 16, x \in \mathbf{R}\}$ | (j) $\{s \mid s^2 > 9, s \in \mathbf{R}\}$ |

2. Describe each of the following in set notation.



In questions 3 to 5, all variables belong to the set of real numbers.

3. Solve and graph each solution set.

(a) $3a - 2 \leq 7$

(b) $3 - 7t > 17$

(c) $5 - 2x > x - 7$

(d) $2(s + 3) - 3(s - 5) \geq 12$

(e) $10 < 5 - (2x - 1)$

(f) $x^2 > 1$

(g) $r^2 \leq 25$

(h) $t + 1 < \frac{3t - 1}{2}$

4. Solve and graph.

(a) $1 \leq x + 3 < 4$

(b) $-1 \leq \frac{x}{3} - 1 < 2$

(c) $4 < \frac{4 - 2b}{-3} < 8$

(d) $2x^2 - 7 \leq 11$

5. Solve.

(a) $\frac{5 - 4b}{3} < 5$

(b) $2 \geq 5 - \frac{2}{3}x \geq 1$

(c) $\frac{3(1 - 2x)}{4} - \frac{1}{2}x < 1$

B

6. A function is defined by $f(x) = 5 - 2x$.

(a) Graph $y = f(x)$.

(b) For $x \leq 1$, determine the range of f graphically.

(c) Verify your answer to (b) algebraically.

(d) For $5 < f(x) \leq 17$, determine the domain of f graphically.

(e) Verify your answer to (d) algebraically.

7. A function is $g(x) = 2(x - 3)^2 + 4$.

(a) Determine the range of g when $x > 4$.

(b) Determine the range of g when $0 \leq x \leq 6$.

(c) Determine the domain of g when $f(x) \leq 12$.

8. There is another way to solve Example 3(a) algebraically. Write an alternative solution.

9. Find the domain and range of f if $-2 < x < 3$ and $f(x) = x(x - 2)$.

10. The function $h(t) = -4.9t(t - 5)$ models the height, h , of a golf ball above the ground in metres, t seconds after the ball is hit. Find the domain of the function.

C

11. Each inequality is written in poor mathematical form. Rewrite each inequality correctly.

(a) $3 < x < -2$

(b) $2 < x > -3$

(c) $5 > x < 2$

12. A parabola with vertex $A(1, 2)$ must pass between $B(5, 10)$ and $C(5, 34)$. Determine the restrictions on the coefficient of x^2 in the equation of the parabola.