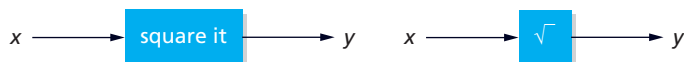


In this section, you will explore the properties of certain inverse functions.

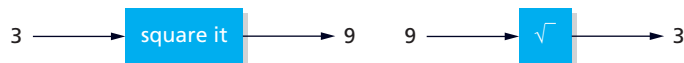
Part 1: Investigating the Relationships

Between $y = \sqrt{x}$ and $y = x^2$

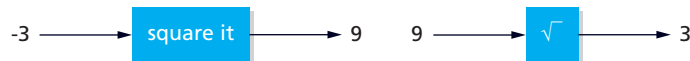
The teacher asked Tom, “What is the inverse of $y = x^2$?” He replied, “ $y = \sqrt{x}$.” Tom explained that the opposite of squaring a value is the taking of the value’s square root. His explanation was correct, but the conclusion, $y = \sqrt{x}$, was not. The input/output diagrams for each function are shown.



These input/output diagrams verify Tom’s answer.



But these input/output diagrams suggest that Tom’s answer is incorrect.



Since point $(-3, 9)$ belongs to the function, then $(9, -3)$ should belong to its inverse. But this point does not satisfy $y = \sqrt{x}$, so this relation cannot be the inverse of $y = x^2$. To get the output of -3 , you could use the equation $y = -\sqrt{x}$.



Then, with an input of 9, the output is $-(3)$ or -3 .



The inverse of $y = x^2$ must include $y = \sqrt{x}$ and $y = -\sqrt{x}$.

Think, Do, Discuss

1. (a) Copy and complete the tables. List the coordinates of points that belong to $y = \sqrt{x}$ in the left table. Use values for x that include 0. List the corresponding points, with the coordinates reversed, for $y = x^2$ in the right table.

x	$y = \sqrt{x}$

x	$y = x^2$

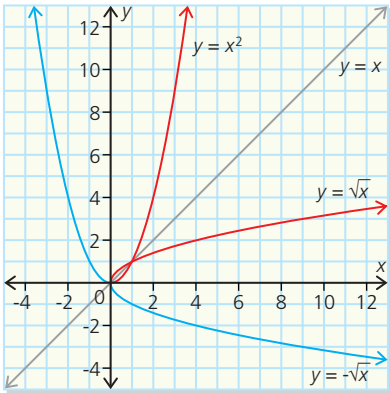
- (b) Graph $y = \sqrt{x}$ and its inverse using the same colour. Use the same scale for each axis, if possible.
2. (a) Copy and complete the tables. List the coordinates of points that belong to $y = -\sqrt{x}$ in the left table. Use values for x that include 0. List the corresponding points, with the coordinates reversed, for $y = x^2$ in the right table.

x	$y = -\sqrt{x}$

x	$y = x^2$

- (b) Graph $y = -\sqrt{x}$ and its inverse using the same colour on the same axes as in step 1(b).
3. Graph $y = x$ on the same axes using a different colour. Your graph should look like this one.
- The red part of $y = x^2$ is called the **right branch** of the parabola, while the blue section is the **left branch**.

4. Fold the graph paper along the line $y = x$, and hold the paper up to the light.
- (a) Which part of the graph is the inverse of $y = \sqrt{x}$? Explain.
- (b) Which part of the graph is the inverse of $y = -\sqrt{x}$? Explain.
- (c) Which part of the graph is the inverse of $y = x^2$? Explain.

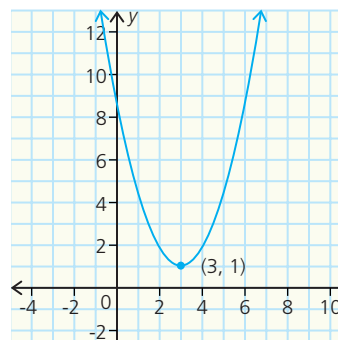


5. State the domain and range of each function. Explain your findings.
- (a) $y = x^2$ (b) $y = \sqrt{x}$ (c) $y = -\sqrt{x}$
6. (a) Does $y = x^2$ define a function? Explain.
 (b) Does $y = \sqrt{x}$ define a function? Explain.
 (c) Does $y = -\sqrt{x}$ define a function? Explain.
 (d) Is the inverse of $y = x^2$ a function? Explain.
 (e) Is the inverse of $y = \sqrt{x}$ a function? Explain.
 (f) Is the inverse of $y = -\sqrt{x}$ a function? Explain.
7. (a) What restriction could you place on the domain of $y = x^2$ so that its inverse is $y = \sqrt{x}$?
 (b) What restriction could you place on the domain of $y = x^2$ so that its inverse is $y = -\sqrt{x}$?
 (c) What is the equation of the inverse of $y = x^2$?
 (d) What restriction could you place on the range of the inverse of $y = x^2$ so that it would be a function, without changing its domain? Explain.
 (e) Is there another restriction that you could place on the range of the inverse of $y = x^2$ so that it would be a function, without changing its domain? Explain.

Part 2: Restricting the Domain of a Quadratic Function So That the Inverse Is a Function

Think, Do, Discuss

1. (a) Set your calculator to degree mode. Then evaluate each of the following.
 i. $\sin 30^\circ$ ii. $\sin 150^\circ$ iii. $\sin 390^\circ$ iv. $\sin 510^\circ$
 (b) For $f(x) = \sin(x)$, all of the ordered pairs $(30^\circ, \blacksquare)$, $(150^\circ, \blacksquare)$, $(390^\circ, \blacksquare)$, and $(510^\circ, \blacksquare)$ belong to f .
 (c) Use your answers from (a) to list ordered pairs that belong to f^{-1} .
 (d) What is the value of $f^{-1}(0.5)$?
 (e) f^{-1} should be the inverse of sine. Use your calculator to evaluate $\sin^{-1}(0.5)$.
 (f) Explain what the calculator has calculated.
2. (a) Sketch any parabola that has vertex $(3, 1)$ and opens up.
 (b) Sketch its inverse.
 (c) How much of the graph of the inverse would you have to erase so that it would pass the vertical line test?
 (d) Redraw the graph of the inverse so that it passes the vertical line test. What is the range of the inverse?
 Stating this inequality places a restriction on the inverse relation to ensure it is a function.



- (e) How would you rewrite the domain of the original function so that its inverse passes the vertical line test?

Stating the new domain with a restriction on the original function guarantees that the inverse is a function.

- (f) State the new domain and range of the function and its inverse.
(g) Why does the vertex help you to see an appropriate restriction?
(h) What other restriction could you place on the domain of the parabola?

3. For $y = 3(x + 1)^2 + 4$,

- (a) state the coordinates of the vertex of this parabola
(b) describe its location and direction of the opening
(c) what restriction could you place on the domain of the function to ensure that the inverse is also a function?

4. (a) What values of y satisfy $y^2 = 16$?

(b) Solve $y^2 = 7$.

(c) Solve $y^2 = x$ for y .

(d) The equation of the inverse of $f(x) = x^2$ is $x = y^2$. Determine f^{-1} .
Why is the inverse not a function?

(e) What restriction could you place on the domain of f so that the inverse is a function? In this case, what is f^{-1} ?

5. (a) Solve $y^2 = 25$.

(b) Solve $(y + 1)^2 = 25$.

(c) Solve $2(y + 1)^2 = 50$.

(d) Solve $2(y + 1)^2 - 3 = 47$.

(e) Solve $2(y + 1)^2 - 3 = x$ for y .

(f) Determine f^{-1} if $f(x) = 2(x + 1)^2 - 3$.

6. (a) Solve $\sqrt{y} = 3$.

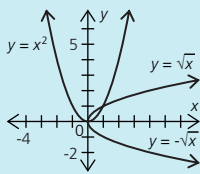
(b) Why does $\sqrt{y} = -3$ have no solution?

(c) What must be true about x if $\sqrt{y} = x$?

(d) Solve for y if $\sqrt{y} = x$.

(e) The inverse of $y = \sqrt{x}$ is $x = \sqrt{y}$. Solve for y .

Key Ideas

- The graphs of $y = x^2$, $y = \sqrt{x}$, and $y = -\sqrt{x}$ are shown.
 
- ♦ The domain of $y = x^2$ is $\mathbf{R}$ and its range is $\{y \mid y \geq 0, y \in \mathbf{R}\}$.
- ♦ The domain of $y = \sqrt{x}$ is $\{x \mid x \geq 0, x \in \mathbf{R}\}$ and its range is $\{y \mid y \geq 0, y \in \mathbf{R}\}$.
- ♦ The domain of $y = -\sqrt{x}$ is $\{x \mid x \geq 0, x \in \mathbf{R}\}$ and its range is $\{y \mid y \leq 0, y \in \mathbf{R}\}$.
- ♦ $y = x^2$ defines a function, but its inverse does not.
- ♦ The inverse of $y = x^2$ is $y = \pm\sqrt{x}$.
- ♦ Both $y = \sqrt{x}$ and $y = -\sqrt{x}$ define functions.
- ♦ The graph of $y = x^2$ reflected in the line $y = x$ produces the graph of $y = \pm\sqrt{x}$.
- Taking the square root of both sides of $y = x^2$ yields a positive square root and a negative square root. Both of these roots are part of the solution. Each square root often describes one branch of a parabola.
- Note the restrictions on the domain and the range of the original function, which may not be given.
- To ensure that the inverse of a function is also a function, you may have to restrict the domain or the range of the original function.
- The vertex of a parabola points to the beginning of a domain or a range. Note the vertex and the direction of opening when you write the restrictions on a domain or range.
- To determine the vertex of a parabola, complete the square.

Example 1

- What is $f^{-1}(x)$ if $f(x) = \sqrt{x}$?
- Graph f and f^{-1} .
- Where do f and f^{-1} intersect?

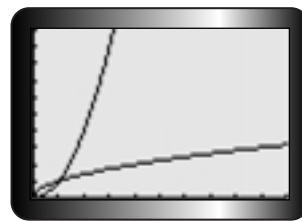
Solution

- The domain and range of $f(x) = \sqrt{x}$ are restricted to real numbers greater than or equal to 0, so the inverse must also have the same restriction. Therefore, $f^{-1}(x) = x^2, x \geq 0$.

Notice that the parabola has only one branch, because the domain is restricted. The parabola would have both branches if x could be any real number.

- (b) Using parametric equations, let $X_{1T} = T$ and $Y_{1T} = \sqrt{T}$ to get the graph of f . To get the graph of f^{-1} , let $X_{2T} = \sqrt{T}$ and $Y_{2T} = T$, with no restriction on the parameter. You can write the inverse as $f^{-1}(x) = x^2$, $x \geq 0$. So let $X_{2T} = T$ and $Y_{2T} = T^2$, but restrict T so that it is greater than or equal to 0. Set **Tmin=0** to restrict T .

Without using parametric equations, graph f by first entering “ $\sqrt{x}$ ” into Y1 and graph f^{-1} by first entering “ $x^2(x \geq 0)$ ” into Y2. Get $\geq$ by pressing **2nd** **MATH** and selecting 4: $\geq$ from the TEST menu.



- (c) In the graph, f and f^{-1} intersect at $(1, 1)$ and at $(0, 0)$. Notice that both points are on the line $y = x$.

Example 2

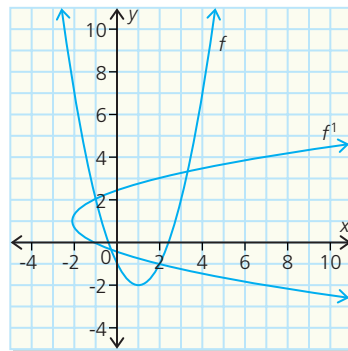
For $f(x) = (x - 1)^2 - 2$,

- (a) graph f and f^{-1} (b) state the domain and range of f
(c) state the domain and range of f^{-1} (d) find the equation of f^{-1}
(e) are f and f^{-1} both functions? Explain.

Solution

- (a) The vertex of f is $(1, -2)$. For $x = 3$ or $x = -1$, $y = 2$, so two other points on the parabola are $(3, 2)$ and $(-1, 2)$. Sketch f using these points. The vertex of f^{-1} is $(-2, 1)$ and two other points on the parabola are $(2, 3)$ and $(2, -1)$. Use these points to sketch f^{-1} .

You can also use *Zap-a-Graph* to graph the parabola $y = (x - 1)^2 - 2$. Select **Reflect in y = x** under the **Transform** menu to get the graph of the inverse.



Another option is to use parametric equations on a graphing calculator to graph both relations.

- Since the vertex is $(1, -2)$, the domain of f is $\mathbf{R}$, and its range is $\{y \mid y \geq -2, y \in \mathbf{R}\}$.
- Interchange the variables to get the inverse. The domain of f^{-1} is $\{x \mid x \geq -2, x \in \mathbf{R}\}$ and its range is $\mathbf{R}$.
- The equation for the original function is $y = (x - 1)^2 - 2$. To find the equation of f^{-1} , interchange x and y and solve for y .

$$x = (\gamma - 1)^2 - 2$$

$$x + 2 = (y - 1)^2$$

$$\pm\sqrt{x+2} = y-1$$

$$\pm\sqrt{x+2}+1=y$$

$\therefore f^{-1}$ is defined by $y = \pm\sqrt{x+2} + 1$

- (e) f is a function because every value of x produces a unique value for y .
However, f^{-1} is not a function because it fails the vertical line test for every value of $x > -2$.

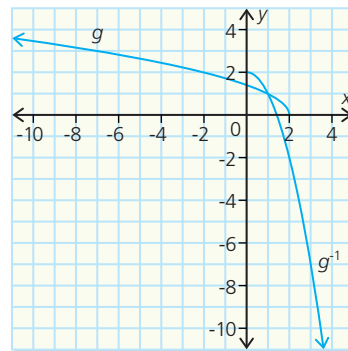
Example 3

For $g(x) = \sqrt{2-x}$,

- (a) graph g and g^{-1} (b) determine the domain and range of g
(c) state the domain and range of g^{-1} (d) find $g^{-1}(x)$

Solution

- (a) Use technology or graph by hand.
- (b) You can get the domain and range of g from the graph, but you can also determine them as follows:
 $\sqrt{2-x}$ must be a real number, so x must be less than or equal to 2.
 $\sqrt{2-x}$ cannot be a negative number, but it can be any other real number. The domain of g is $\{x \mid x \leq 2, x \in \mathbf{R}\}$ and its range is $\{y \mid y \geq 0, y \in \mathbf{R}\}$.
- (c) Interchange the variables to get the inverse. Then the domain of g^{-1} is $\{x \mid x \geq 0, x \in \mathbf{R}\}$ and its range is $\{y \mid y \leq 2, y \in \mathbf{R}\}$.
- (d) g^{-1} is the right branch of a parabola that has vertex $(0, 2)$ and opens down.
The equation of g^{-1} is $y = -x^2 + 2$, but x must be greater than or equal to 0.
The restriction yields only the right branch of the parabola. Therefore,
 $g^{-1}(x) = -x^2 + 2, x \geq 0$.



Example 4

Dean was asked to find the inverse of $f(x) = 3(x-1)^2 - 2$, and he wrote

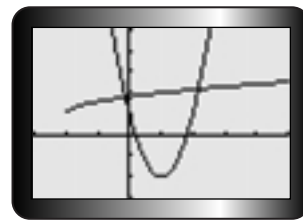
$$y = \sqrt{\frac{x+2}{3}} + 1.$$

- (a) Use graphs to help explain Dean's answer. Is the answer correct?
- (b) What other function should you graph to complete the graph of f^{-1} ?
- (c) Is f^{-1} a function? Explain.
- (d) What is the correct equation for f^{-1} ?
- (e) Determine a restriction on the range of the inverse to ensure that f^{-1} is also a function. Do not change the domain.

Solution

- (a) Using a TI-83 Plus calculator, press **MODE** and make sure that the graphing mode is **Func**.

To graph $f(x) = 3(x - 1)^2 - 2$ and $f^{-1}(x) = \sqrt{\frac{x+2}{3}} + 1$, enter the equations into **Y1** and **Y2**, respectively, in the equation editor. The graph of f is a parabola, but Dean's answer for f^{-1} represents only one branch of a parabola. This relation is not the inverse of f .



- (b) Once $y = -\sqrt{\frac{x+2}{3}} + 1$ is graphed, then the graph of the inverse is complete.
- (c) $y = \pm\sqrt{\frac{x+2}{3}} + 1$ is not a function, even though it is the inverse of $f(x) = 3(x - 1)^2 - 2$. Dean's answer is a function because the relation passes the vertical line test.
- (d) The correct equation for the inverse is $y = \pm\sqrt{\frac{x+2}{3}} + 1$. The graph of this function includes both branches of the parabola.
- (e) For the inverse, only one branch of the parabola can be a function. Then $y \geq 1$, or $y \leq 1$. The domain of the inverse is $\{x \mid x \geq -2, x \in \mathbf{R}\}$. To make the inverse a function, without changing the domain, simply choose one of the branches of the parabola. For $y \geq 1$, the domain is still $\{x \mid x \geq -2, x \in \mathbf{R}\}$. The range is $\{y \mid y \geq 1, y \in \mathbf{R}\}$, and f^{-1} is a function.

Example 5

For $f(x) = 3x^2 + 6x + 5$,

- (a) determine the vertex of the parabola
- (b) restrict the domain so that f^{-1} is a function
- (c) find $f^{-1}(x)$ using that restriction

Solution

- (a) Complete the square to find the vertex.

$$f(x) = 3x^2 + 6x + 5$$

$$= 3(x^2 + 2x) + 5$$

$$= 3(x^2 + 2x + 1 - 1) + 5$$

$$= 3(x^2 + 2x + 1) - 3 + 5$$

$$= 3(x + 1)^2 + 2$$

Factor the coefficient of x^2 from the terms with variables.

Add and subtract the square of half the coefficient of x .

To remove the last constant from the brackets, multiply the last constant by the coefficient.

Factor the perfect square and simplify.

The vertex of the parabola is $(-1, 2)$.

- (b) The parabola opens up. To ensure that the inverse is also a function, choose one branch of the parabola. Then the restriction on the domain is $x \geq -1$.

- (c) The equation of f is $y = 3(x + 1)^2 + 2$ and $x \geq -1$.

To get the equation for f^{-1} , interchange the variables and solve for y . The range is $y \geq -1$.

$$\therefore x = 3(y + 1)^2 + 2 \quad \text{Solve for } y.$$

$$x - 2 = 3(y + 1)^2$$

$$\frac{x - 2}{3} = (y + 1)^2$$

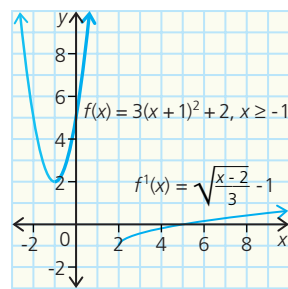
$$\pm \sqrt{\frac{x - 2}{3}} = y + 1$$

But $y \geq -1$, so

$$\sqrt{\frac{x - 2}{3}} = y + 1$$

$$y = \sqrt{\frac{x - 2}{3}} - 1$$

$$\therefore f^{-1}(x) = \sqrt{\frac{x - 2}{3}} - 1$$

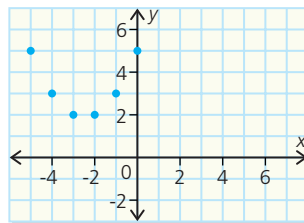


Practise, Apply, Solve 3.6

A

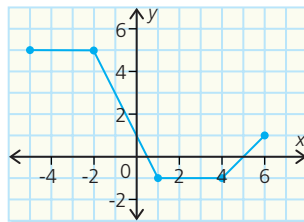
1. The graph of f is shown.

- (a) Graph f^{-1} .
 (b) Write a subset of f^{-1} that has the same domain as f^{-1} , but represents a function.



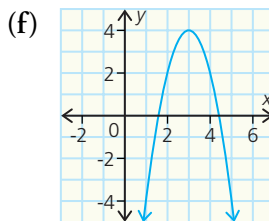
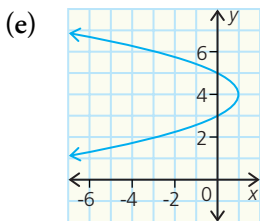
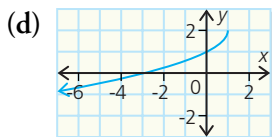
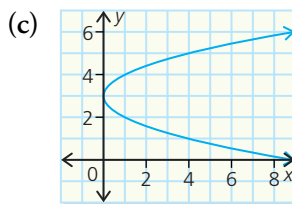
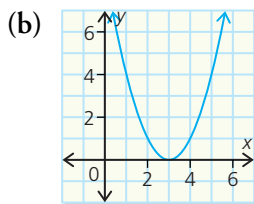
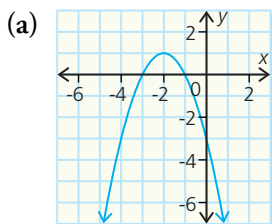
2. The graph of g is shown.

- (a) Graph g^{-1} .
 (b) State the domain and range of g .
 (c) State the domain and range of g^{-1} .
 (d) What restriction could you place on the domain of the function so that the inverse is a function and has the same domain as g^{-1} ?



3. For each relation,

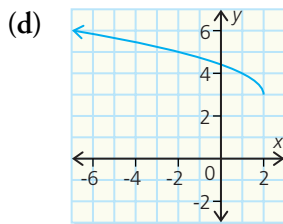
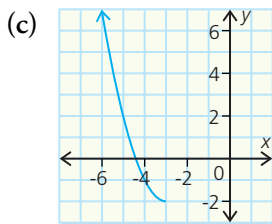
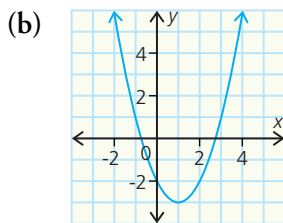
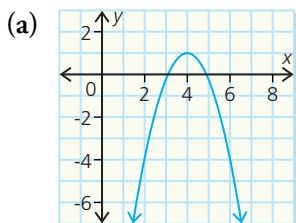
- state restrictions on the domain or range of the relation that make it a function
- graph the inverse relation
- if necessary, restate the restrictions on the domain or range of the original relation to make its inverse a function



B

4. Each graph shows a function f that is a parabola or a branch of a parabola.

- Determine $f(x)$.
- Graph f^{-1} .
- State restrictions on the domain or range of f to make its inverse a function.
- Determine the equations for f^{-1} .



5. (a) Sketch the graph of $f(x) = -2(x + 1)^2 + 3$ by hand.
 (b) Sketch the graph of its inverse.
 (c) Restrict the domain of f so that both f and its inverse are functions.
6. (a) Graph $g(x) = -\sqrt{x}$ by hand.
 (b) Graph its inverse on the same axes.
 (c) State the domain and range of g^{-1} .
 (d) Determine the equation for g^{-1} .
7. (a) Graph $g(x) = 3 + \sqrt{x}$ using parametric equations on the TI-83 Plus.
 (b) Graph its inverse.
 (c) Determine the equation for g^{-1} .
 (d) Use your expression for g^{-1} to graph its inverse.
 (e) Compare your graphs from (b) and (d). What restriction did you need to place on T to ensure there are no “extra” points?
8. (a) Graph $f(x) = \sqrt{x - 2}$ using parametric equations on the TI-83 Plus.
 (b) Graph its inverse.
 (c) Determine the equation for f^{-1} .
 (d) Use your expression for f^{-1} to graph the inverse.
 (e) Compare your graphs from (b) and (d). What restriction did you need to place on T to ensure there are no “extra” points?
9. Given $f(x) = 7 - 2(x - 1)^2$, $x \geq 1$. Graph f and f^{-1} , and then determine

(a) $f(3)$	(b) $f^{-1}(x)$	(c) $f^{-1}(5)$
(d) $f^{-1}(2a + 7)$	(e) $7 - 2(f^{-1}(x) - 1)^2$	
10. **Knowledge and Understanding:** For $f(x) = -x^2 + 6x + 5$,
 - (a) determine the vertex of the parabola
 - (b) restrict the domain so that f^{-1} is a function
 - (c) find f^{-1} using that restriction
11. For $-2 < x < 3$ and $f(x) = 3x^2 - 6x$, determine
 - (a) the domain and range of f
 - (b) f^{-1}
12. Graph $f(x) = -\sqrt{x} + 5$ and determine
 - (a) the domain and range of f
 - (b) f^{-1}
13. Describe the relation between $y = x^2$ and $y = \sqrt{x}$.

14. The height of a golf ball after Lori Kane hits it is shown in the table below.

Time (s)	0	0.5	1	1.5	2	2.5
Height (m)	0	12.375	22.5	30.375	36.0	39.375



- (a) Use first differences and second differences to extend the table.
 - (b) Graph the data and the curve of best fit for the relationship.
 - (c) Graph the inverse relation and its curve of best fit.
 - (d) Is the inverse a function? Explain.
15. **Communication:** A student writes, “The inverse of $y = -\sqrt{x+2}$ is $y = x^2 - 2$.” Explain why this statement is not true.
16. Do you have to restrict either the domain or the range of the function $y = \sqrt{x+2}$ to make its inverse a function? Explain.
17. **Application:** In section 3.4, question 20, each digit of a credit card number, the input, was coded by subtracting the digit from 9. Each input, x , is any single digit, so $0 \leq x \leq 9$. The range is the set of integers between 0 and 9. This code is very convenient, but easy to decode. A student developed a different code: “Multiply the input digit by a number that is 2 less than the digit.” Let f be the coding function.
- (a) Determine $f(x)$.
 - (b) Determine the equation for f^{-1} .
 - (c) Is f^{-1} a function? Explain.
 - (d) Suppose the input digits are restricted to the integers between 2 and 5. What is the range of f ?
 - (e) Suppose that the output digits must be between 1 and 16. What restrictions should be placed on the input digits?
 - (f) What restrictions would you place on both f and f^{-1} so that they are functions? (**Hint:** There can be as many digits as possible.) Why is this restriction important?
18. **Check Your Understanding:** Given the graph of a function, how would you graph its inverse? How can you ensure that its inverse is a function?

C

- 19. Thinking, Inquiry, Problem Solving:** The meat department manager discovers he could sell $m(x)$ kg of ground beef in a week, where $m(x) = 14\,700 - 3040x$, if he sold it at $\$x/\text{kg}$. He pays his supplier $\$3.21/\text{kg}$ for the beef.
- Determine an algebraic expression for $P(x)$, where P represents the total profit from selling ground beef for one week.
 - Find the equation for P^{-1} .
 - Write an expression in function notation that represents the price that will earn $\$1900$ in profit, and then evaluate. Explain.
 - Determine the price that will maximize profit.
 - The supply cost drops to $\$3.10/\text{kg}$. What price should the manager set? How much profit will be earned at this price?
 - Create a table by recording the optimal profit for several different supply costs.
 - Why might a change in supply cost also change the volume of sales?
- 20.** Find the range of $f(x) = \frac{1}{x}$, where $-2 < x < 3$.



The Chapter Problem—Cryptography

Apply what you have learned about inverse functions to answer these questions about the Chapter Problem on page 218.

- CP11.** What is the restriction on the domain of the coding function in the Chapter Problem on page 218? Why?
- CP12.** What is the restriction on the range of the coding function? Why?
- CP13.** What was there about the coding function that guaranteed the range would work out appropriately?
- CP14.** A student suggested this code: instead of multiplying by 17, square the number, divide by 26, and add 1 to the remainder.
- Graph this function.
 - Why would this code not work?
 - What is necessary for any code to work?