



3.7

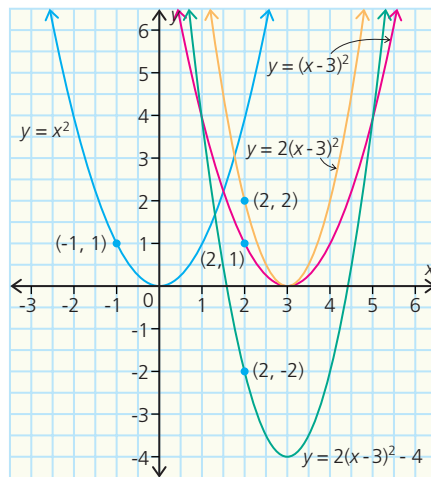
Transformations and Function Notation

In grade 10, you used transformations to sketch the graph of a quadratic relation such as $y = 2(x - 3)^2 - 4$. First graph the curve $y = x^2$ and then translate this graph 3 units to the right, stretch it vertically by a factor of 2, and then translate it 4 units down.

This graph shows the effect of these transformations on point $(-1, 1)$. The horizontal translation of $+3$ takes the point to $(2, 1)$. The vertical stretch of factor 2 takes it to $(2, 2)$. Finally the vertical translation of -4 takes it down to $(2, -2)$.

Applying these transformations to all the points that lie on the graph of $y = x^2$ results in the graph of $y = 2(x - 3)^2 - 4$.

In this lesson, you will investigate how you can transform the graph of any function $y = f(x)$ to obtain the graph of any function in the form $y = af[k(x - p)] + q$.

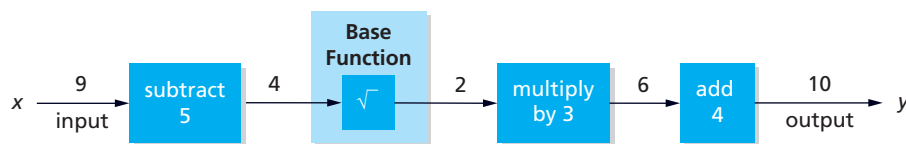


Part 1: The Graph of $y = af(x - p) + q$ Compared to the Graph of $y = f(x)$

Think, Do, Discuss

1. Using a TI-83 Plus, press **ZOOM** **6** to set the standard window.
2. (a) Graph the base function $y = \sqrt{x}$. Enter this function into **Y1** in the equation editor.
(b) What is the domain? What is the range?
(c) What transformation would you apply to the graph of $y = \sqrt{x}$ to obtain the graph of $y = \sqrt{x - 5}$? Sketch your prediction on graph paper.
(d) Graph $y = \sqrt{x - 5}$ using a graphing calculator to check your answer. Enter this function into **Y2** in the equation editor.
(e) What is the domain of $y = \sqrt{x - 5}$? Explain. What is its range? Explain.
(f) One point on the graph of $y = \sqrt{x}$ is $(4, 2)$. What is the corresponding point on the graph of $y = \sqrt{x - 5}$? Explain the transformation.
(g) Evaluate $y = \sqrt{x - 5}$ for $x = 9$. How does this answer correspond to your answer to (f)?

3. (a) What transformations would you apply to the graph of $y = \sqrt{x}$ to obtain the graph of $y = 3\sqrt{x - 5}$? Sketch your prediction on graph paper.
- (b) Graph $y = 3\sqrt{x - 5}$ using a graphing calculator to check your answer. Enter this function into **Y3** in the equation editor.
- (c) What is the domain of $y = 3\sqrt{x - 5}$? Explain. What is its range? Explain.
- (d) One point on the graph of $y = \sqrt{x}$ is (4, 2). What is the corresponding point on the graph of $y = 3\sqrt{x - 5}$? Explain the transformation.
- (e) Evaluate $y = 3\sqrt{x - 5}$ for $x = 9$. How does this correspond to your answer to (e)?
4. (a) What transformations would you apply to the graph of $y = \sqrt{x}$ to obtain the graph of $y = 3\sqrt{x - 5} + 4$? Sketch your prediction on graph paper.
- (b) Graph $y = 3\sqrt{x - 5} + 4$ using the graphing calculator to check your answer. Enter this function into **Y4** in the equation editor.
- (c) What is the domain of $y = 3\sqrt{x - 5} + 4$? Explain. What is its range? Explain.
- (d) One point on the graph of $y = \sqrt{x}$ is (4, 2). What is the corresponding point on the graph of $y = 3\sqrt{x - 5} + 4$? Explain the transformation.
- (e) Evaluate $y = 3\sqrt{x - 5} + 4$ for $x = 9$. How does this answer correspond to your answer to (d)?
- (f) The “base” function in all of these examples is the square root. In (e), what operation did you do before taking the square root? Which transformation did this operation correspond to? Was this transformation vertical or horizontal?
- (g) In (e), what did you do after finding the square root? Which transformations did these operations correspond to? Were the transformations vertical or horizontal?
- (h) The input/output diagram shows how to evaluate $y = 3\sqrt{x - 5} + 4$ for $x = 9$. On this diagram, which operations correspond to vertical transformations? to horizontal transformations?



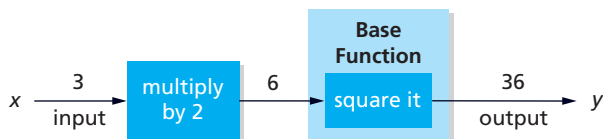
- (i) Notice that the coordinates of point (4, 2) appear in the diagram above. The 4 is to the left of the base function and the 2 is to the right of the base function. Why does this make sense?
- (j) Move to the right of the base function in this diagram. What is the order of the operations? In what order should you apply the transformations?

- (k) Move to the left of the base function in this diagram. What is the operation? What transformation does this operation correspond to?
- (l) Find other values of x that would produce integer values of y . How can you get integral answers for y without guessing?
5. Using graphing technology, explore the effects of varying a , p , and q in $y = a\sqrt{x-p} + q$ on the graph of $y = \sqrt{x}$. For each graph, record the following in your notebook:
- the equation
 - the values of a , p , and q
 - the domain and range of the function
 - a description of the transformations that must be applied to the graph of $y = \sqrt{x}$ to obtain the graph of $y = a\sqrt{x-p} + q$
6. (a) Use graphing technology to graph $y = x^3$. What is the domain? What is the range? Is this relation a function?
- (b) Explore the effects of varying a , p , and q in $y = a(x-p)^3 + q$ on the graph of $y = x^3$. For each graph, record the following in your notebook:
- the equation
 - the values of a , p , and q
 - the domain and range of the function
 - a description of the transformations that must be applied to the graph of $y = x^3$ to obtain the graph of $y = a(x-p)^3 + q$

Part 2: The Graph of $y = f(kx)$ Compared to $y = f(x)$

Think, Do, Discuss

- (a) Clear all the equations from the equation editor. Graph the function base $y = x^2$. Enter this equation into Y1. What is the domain? What is the range?
- (b) On the same set of axes, graph $y = (2x)^2$. Enter this equation into Y2. How does the new graph compare to the graph of $y = x^2$? What is the domain? What is the range?
- (c) The input/output diagram shows how to evaluate $y = (2x)^2$ for $x = 3$. What is the corresponding point on the graph of $y = x^2$? Explain.



- (d) Create a table for $y = (2x)^2$. Create another table of corresponding points for $y = x^2$.

x	$y = (2x)^2$
3	36
0	
1	
2	
4	
5	

... corresponds to ...

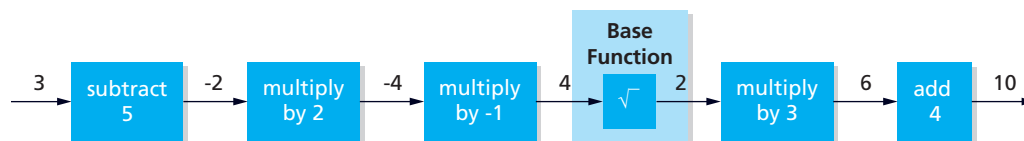
x	$y = x^2$
6	36

- (e) Suppose that you had started with the table for $y = x^2$. Describe how you would find the corresponding points on the graph of $y = (2x)^2$.
This transformation is called a horizontal stretch of factor $\frac{1}{2}$.
- (f) Press **[2nd]** **[GRAPH]** to examine the table for both graphs. Is your work for (d) and (e) correct?
- (g) Explore what happens to the graph and table when you change the coefficient k in $y = (kx)^2$ from 2 to some other number. Try numbers between 0 and 1, greater than 1, and less than 0.
2. Explore the effect of varying k in $y = \sqrt{kx}$ on the graph of $y = \sqrt{x}$.
For each graph, record the following:
- the equation
 - the value of k (use numbers between 0 and 1, greater than 1, and less than 0)
 - an input/output diagram for the equation
 - a table for $y = \sqrt{x}$ and the corresponding points for $y = \sqrt{kx}$
 - the domain and range of the function
 - a description of the transformations that must be applied to $y = \sqrt{x}$ to obtain the graph of $y = \sqrt{kx}$
3. Explore the effect of varying k in $y = (kx)^3$ on the graph of $y = x^3$.
For each graph, record the following:
- the equation
 - the value of k
 - an input/output diagram for the equation
 - a table for $y = x^3$ and the corresponding points for $y = (kx)^3$
 - the domain and range of the function
 - a description of the transformations that must be applied to $y = x^3$ to obtain the graph of $y = (kx)^3$

Part 3: The Graph of $y = af[k(x - p)] + q$ Compared to $y = f(x)$

Think, Do, Discuss

1. (a) Clear all the equations from the equation editor. Graph the function $y = \sqrt{x}$. Enter this equation into Y1.
- (b) The input/output diagram shows how to evaluate $y = 3\sqrt{-2(x - 5)} + 4$ for $x = 3$. What is the corresponding point on the graph of $y = \sqrt{x}$? Explain.



- (c) Notice that the coordinates of point (4, 2) appear in this diagram. The 4 is to the left of the base function and the 2 is to the right of the base function. Why does this make sense?
- (d) Move to the right of the base function in this diagram. What is the order of the operations? In what order should you apply the transformations?
- (e) Move to the left of the base function in this diagram. What is the order of the operations? What transformations do these operations correspond to? In what order should you apply these transformations?
- (f) Find other values of x that would produce integer values of y . How can you get integral answers for y without guessing?
- (g) What transformations are applied to the graph of $y = \sqrt{x}$ to obtain the graph of $y = 3\sqrt{-2(x - 5)} + 4$? Sketch your prediction on graph paper.
- (h) Graph $y = 3\sqrt{-2(x - 5)} + 4$ using a calculator to check your answer.
- (i) What is the domain of $y = 3\sqrt{-2(x - 5)} + 4$? What is its range? Explain.

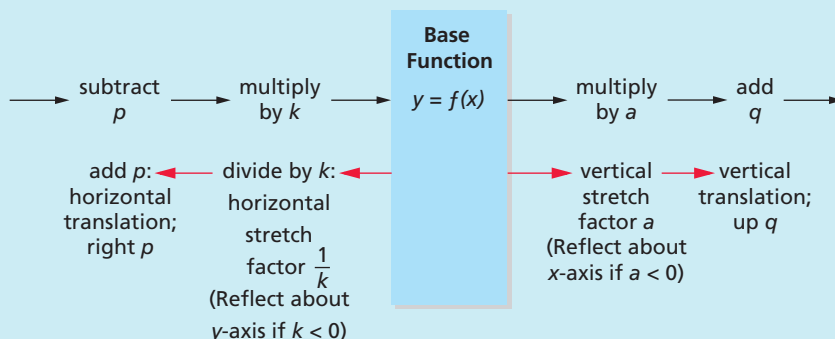
Focus 3.7

Key Ideas

- To graph $y = af[k(x - p)] + q$ from the graph of $y = f(x)$, consider the following:
 - ♦ The value of a determines the vertical stretch and whether there is a reflection in the x -axis or not. For $a > 0$, the graph of $y = f(x)$ is stretched vertically by factor a . For $a < 0$, the graph is reflected in the x -axis and stretched vertically by factor a .
 - ♦ The value of k determines the horizontal stretch and whether there is a reflection in the y -axis or not. For $k > 0$, the graph of $y = f(x)$ is stretched horizontally by factor $\frac{1}{k}$. For $k < 0$, the graph is reflected in the y -axis and stretched horizontally by factor $\frac{1}{k}$.

- ♦ For $p > 0$ and $q > 0$, the graph is translated p units to the right and q units up. For $p < 0$ and $q < 0$, the graph is translated p units to the left and q units down.
- ♦ The input/output diagram shows the sequence and type of transformations for transforming the graph of $y = f(x)$ into the graph of $y = af[k(x - p)] + q$.

$$y = af[k(x - p)] + q$$

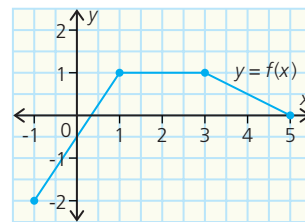


The red arrows show the order for applying the transformations to the base function.

Note that arranging the equation so that the multiplication operation is immediately before and after the base functions allows us to perform all stretches and flips before translations.

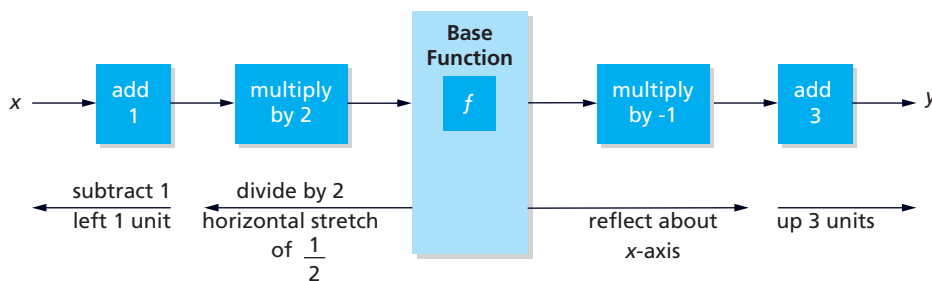
Example 1

- Draw an input/output diagram to show the order of operations for the function $y = -f[2(x + 1)] + 3$.
- Sketch its graph on the same axes as $y = f(x)$.
- Create a table to verify that your graph is correct.
- State the domain and range of the functions $y = f(x)$ and $y = -f[2(x + 1)] + 3$.

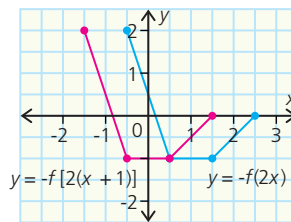
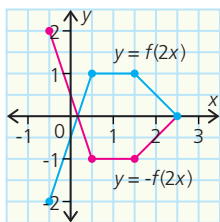
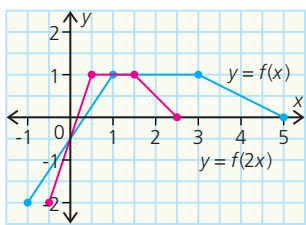


Solution

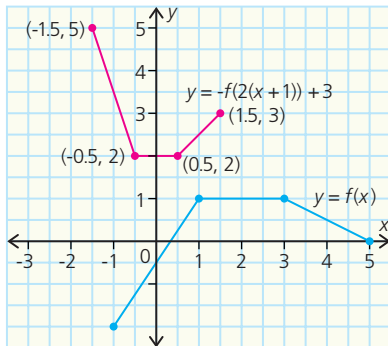
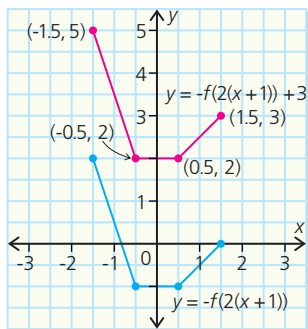
(a)



- (b) The graphs show the results after each transformation is applied. The horizontal stretch of factor $\frac{1}{2}$ gives $y = f(2x)$. A reflection in the x -axis gives $y = -f(2x)$. A horizontal translation of -1 results in $y = -f[2(x + 1)]$.



A vertical translation of 3 gives the graph of $y = -f[2(x + 1)] + 3$. The final graph shows $y = f(x)$ and $y = -f[2(x + 1)] + 3$ on the same axes.



(c)

x	Add 1. $x + 1$	Multiply by 2. $2(x + 1)$ original x	Apply f . $f[2(x + 1)]$ original y	Multiply by -1 . $-f[2(x + 1)]$	Add 3. $y = -f[2(x + 1)] + 3$
-1.5	-0.5	-1	$f(-1) = -2$	2	5
-0.5	0.5	1	$f(1) = 1$	-1	2
0.5	1.5	3	$f(3) = 1$	-1	2
1.5	2.5	5	$f(5) = 0$	0	3
0	1	2	$f(2) = 1$	-1	2
1	2	4	$f(4) = 0.5$	-0.5	2.5

The table shows that $(-1.5, 5)$, $(-0.5, 2)$, $(0.5, 2)$, $(1.5, 3)$, $(0, 2)$, and $(1, 2.5)$ are all points on the new graph.

- (d) For $y = f(x)$, the domain is $\{x \mid -1 \leq x \leq 5, x \in \mathbf{R}\}$ and the range is $\{y \mid -2 \leq y \leq 1, y \in \mathbf{R}\}$. From the graph of $y = -f[2(x + 1)] + 3$, see that the domain is $\{x \mid -1.5 \leq x \leq 1.5, x \in \mathbf{R}\}$ and the range is $\{y \mid 2 \leq y \leq 5, y \in \mathbf{R}\}$.

Notice that without the graph you could still determine the domain and range of the function by applying the transformations as follows:

Begin with the domain of $y = f(x)$: $-1 \leq x \leq 5$.

After a horizontal compression of factor $\frac{1}{2}$: $-0.5 \leq x \leq 2.5$

After a translation of 1 unit left: $-1.5 \leq x \leq 1.5$.

The vertical transformations do not affect the domain.

For the range of $y = f(x)$: $-2 \leq y \leq 1$

After a reflection in the x -axis: $2 \geq y \geq -1$ or $-1 \leq y \leq 2$

After a translation of 3 units up: $2 \leq y \leq 5$

The horizontal transformations do not affect the range.

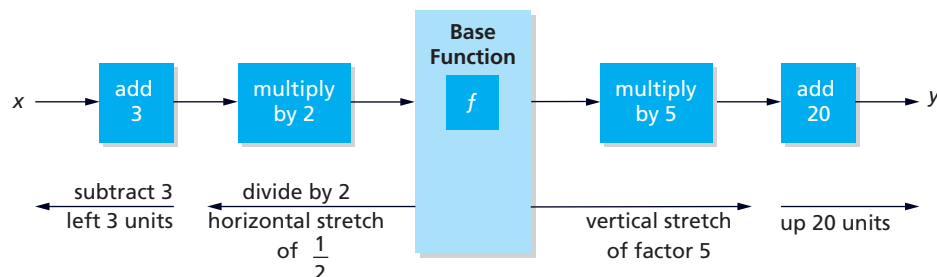
Example 2

Describe the sequence of transformations for creating the graph of $y = 5[f(2x + 6) + 4]$ from the graph of $y = f(x)$.

Solution

Rearrange $y = 5[f(2x + 6) + 4]$ to the equivalent form $y = 5f[2(x + 3)] + 20$.

The input/output diagram for this function is shown.



Begin by stretching $y = f(x)$ horizontally by factor $\frac{1}{2}$ and vertically by factor 5.

Then translate the graph 3 units to the left and 20 units up.

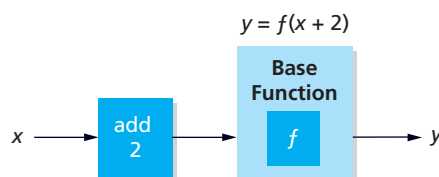
Practise, Apply, Solve 3.7

A

- Communication:** Comment on this statement.

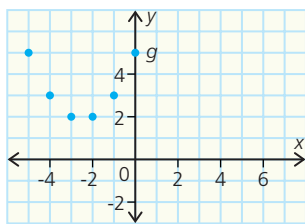
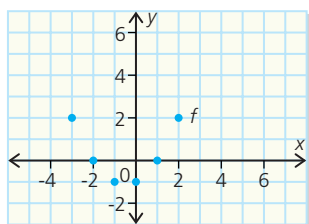
The basic shape of the final graph after transformations is the same as the shape of the original graph.

- The ordered pairs (1, 5), (2, 3), and (3, 7) belong to a function f . The input/output diagram for a function g is shown. State the coordinates of three points on g .

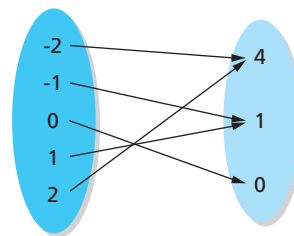


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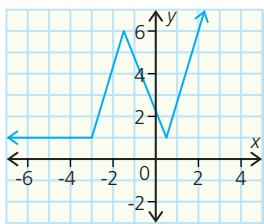
3. Given $g = \{(1, 2), (2, 5), (3, 7), (4, 8)\}$, list the ordered pairs that belong to the function.
- (a) $y = 2g(x)$ (b) $y = g(x) + 2$ (c) $y = g(x + 2)$ (d) $y = g(2x)$
4. $f = \{(0, 3), (1, 1), (2, 4), (3, -1)\}$. After two transformations, the new function is $g = \{(0, 0), (5, -2), (10, 1), (15, -4)\}$.
- (a) Graph f and g .
- (b) Write one possible sequence of transformations that were applied to f to give g .
- (c) Draw an input/output diagram to show the sequence of the operations that result in g .
- (d) Write an equation for $g(x)$ in terms of f .
- (e) Write a sequence of two transformations to change g into f .
- (f) Draw an input/output diagram to show the sequence of the operations to change g into f .
- (g) Write an equation for $f(x)$ in terms of g .
5. The graph of f is on the left and the graph of g is on the right.
- (a) Find the sequence of transformations that were applied to the graph of f to get the graph of g .
- (b) Draw an input/output diagram to show the sequence of transformations.
- (c) Write an equation for $g(x)$ in terms of f .



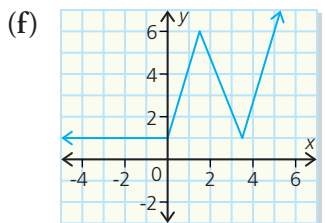
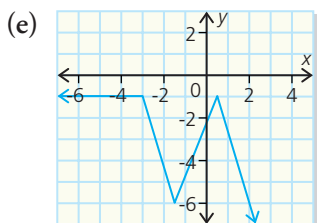
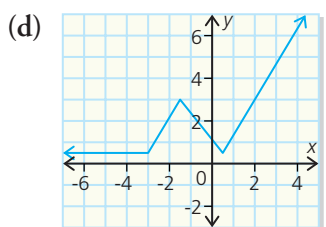
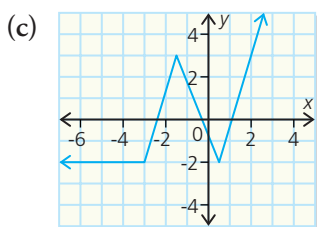
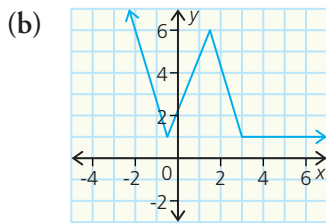
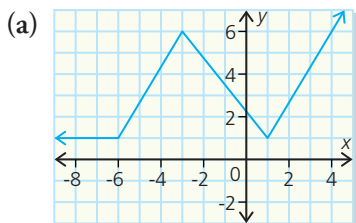
6. f is illustrated by the arrow diagram.
- (a) Draw an input/output diagram for $y = 2f(x - 3)$.
- (b) List the transformations in the correct sequence that you would apply to f to get $y = 2f(x - 3)$.
- (c) Graph f and $y = 2f(x - 3)$ on the same axes.



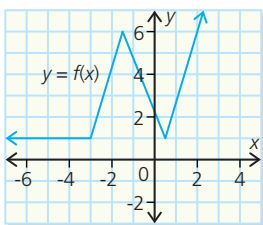
7. The graph of $y = f(x)$ is shown. Match the correct equation to each graph. Justify your choices.



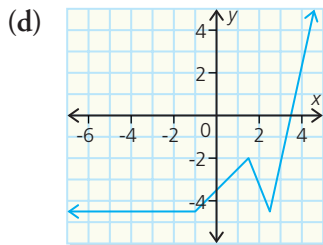
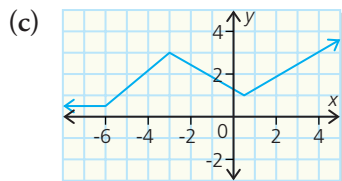
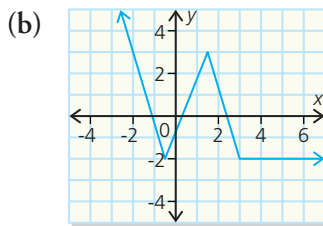
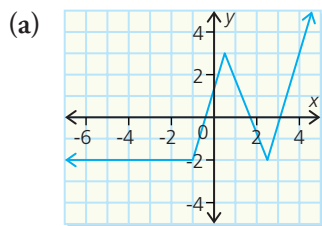
- | | |
|--------------------|---------------------|
| i. $y = f(x - 3)$ | ii. $y = f(0.5x)$ |
| iii. $y = -f(x)$ | iv. $y = f(-x)$ |
| v. $y = f(x + 3)$ | vi. $y = f(2x)$ |
| vii. $y = 2f(x)$ | viii. $y = 0.5f(x)$ |
| ix. $y = f(x) + 3$ | x. $y = f(x) - 3$ |



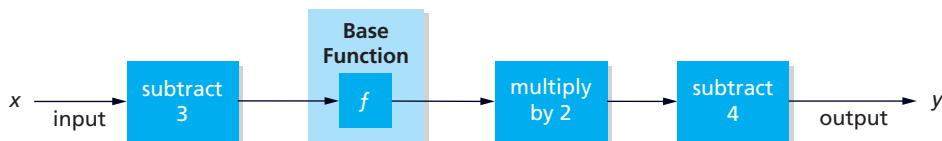
8. The graph of $y = f(x)$ is shown. Match each equation to its corresponding graph. Justify your choices.



- | | |
|--------------------------------|---|
| i. $y = 0.5f(2x)$ | ii. $y = 2f(0.5x)$ |
| iii. $y = 0.5f(0.5x)$ | iv. $y = 2f(2x)$ |
| v. $y = 2f(x + 3)$ | vi. $y = f(2x) + 3$ |
| vii. $y = 3f(x + 2)$ | viii. $y = f(x + 3) + 2$ |
| ix. $y = f(x - 2) - 3$ | x. $y = f(x + 2) + 3$ |
| xi. $y = -f(x) - 3$ | xii. $y = f(-x) - 3$ |
| xiii. $y = -f(x + 3)$ | xiv. $y = f(-x - 3)$ |
| xv. $y = 0.5f(3x + 5)$ | xvi. $y = 2f\left(\frac{x}{3} - 2\right) - 5$ |
| xvii. $y = 0.5f[3(x - 2)] - 5$ | xviii. $y = 0.5f[3(x + 2)] + 5$ |

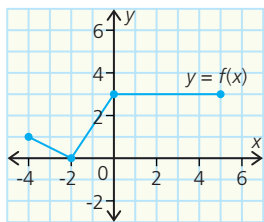


9. i. Graph $f(x) = \frac{1}{x}$ using graphing technology.
 ii. Describe the transformations that you would apply to the graph of f to transform f into each of the following.
- (a) $y = \frac{1}{x-2}$ (b) $y = \frac{1}{x+1}$ (c) $y = \frac{1}{x} + 2$
 (d) $y = \frac{1}{x} - 6$ (e) $y = 0.5\left(\frac{1}{x}\right)$ (f) $y = \frac{2}{x}$
 (g) $y = \frac{1}{0.5x}$ (h) $y = \frac{1}{2x}$ (i) $y = -\frac{1}{x}$
- iii. Verify your answers in ii with graphing technology.
 iv. State the domain and range of each function in ii.
 v. Why are the graphs of (f) and (g) the same?
10. Consider $g(x) = x^2$.
- (a) Draw an input/output diagram for $y = -3g(x-2) + 1$.
 (b) A point on the graph of g is $(4, 16)$, what is the corresponding point on the transformed function?
 (c) List the transformations in the sequence that you would apply to the graph of $y = g(x)$ to graph $y = -3g(x-2) + 1$.
 (d) State the domain and range of the transformed function. Justify your answer.
11. The input/output diagram shows function g in terms of function f .



- (a) Evaluate $g(4)$ if $f(x) = x^3$.
 (b) Point $(2, 0) \in f$. What point belongs to g ?
 (c) Graph g if $f(x) = x^2$.

12. The graph of $g(x) = \sqrt{x}$ is reflected in the y -axis, stretched vertically by factor 3, and then translated 5 units right and 2 units down.
- Draw the graph of the new function.
 - Write the equation of the new function.
13. The graph of $f(x) = x^3$ is translated left 5 units and up 3 units.
- Use a graphing calculator to graph f .
 - Determine the equation of the translated function. Verify your answer by graphing your equation.
14. The graph of $y = f(x)$ is reflected in the y -axis, stretched vertically by factor 3, and then translated up 2 units and 1 unit to the left.
- Draw the input/output diagram.
 - Write the equation for the new function in terms of f .
15. **Knowledge and Understanding:** Given the graph of $y = f(x)$, draw the graph of $y = 2f(-3x + 6) + 2$.



16. **Application:** A function f has domain $\{x \mid x \geq -4, x \in \mathbf{R}\}$ and range $\{y \mid y < -1, y \in \mathbf{R}\}$. Determine the domain and range of each function.
- $y = 2f(x)$
 - $y = f(-x)$
 - $y = 3f(x + 1) + 4$
 - $y = -2f(-x + 5) + 1$
17. **Thinking, Inquiry, Problem Solving:** A squash ball was dropped from different heights. The height and the time taken for the ball to hit the ground was recorded.

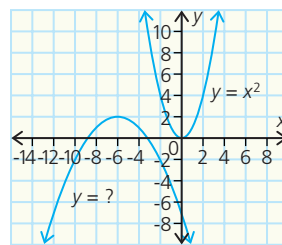
Height (m)	Time (s)
1.5	0.7
2	0.8
2.5	0.86
3	0.93
1	0.6

- Create a scatter plot for the data.
- Find the equation in the form $y = a\sqrt{x}$ that most closely fits the data.
- What other transformation might be needed to make the graph fit the data more closely?
- Modify your equation until it fits the data as closely as possible.

- 18.** Three transformations are applied to $y = x^2$: a vertical stretch of factor 2, a translation of 3 units right, and a translation of 4 units down.
- Is the order of the transformations important?
 - Is there any other sequence of these transformations that could produce the same result?
- 19.** The graph of $y = f(x)$ is transformed into $y = 3f(2x - 4)$.
- Describe the transformations in the correct sequence.
 - Describe another sequence of three transformations that would produce the same final graph.
- 20.** The transformations in Example 1 were performed in the following order: horizontal stretch factor $\frac{1}{2}$, reflection in the x -axis, translation 1 unit left, and then translation 3 units up.
- List all the possible sequences of these transformations that would produce the same final graph from the original function. Explain.
 - Are there any other sequences of different transformations that would produce the same final graph from the original function? Explain.
- 21. Check Your Understanding:** Describe how you determine which transformations are needed to graph a function that is expressed in terms of another function.

C

- 22.** The graphs of $y = x^2$ and another parabola are shown.
- Determine a combination of transformations that would produce the second parabola from the first.
 - Determine a possible equation for the second parabola.



- 23.** Ari recorded the following data from a science experiment.

Length (cm)	3.1	3.5	4	4.5	5	5.5	6
Time (s)	0.6	3.4	5.3	5.9	6.8	7.6	8.3

- When he graphed the data, he found that the graph resembled the graph of $y = \sqrt{x}$. Determine the equation to model the data by trial and error on the graphing calculator.
- Determine an algebraic model by first finding the equation for the inverse function.
- Compare your answers from (a) and (b).

- 24.** A quadratic function has vertex $(3, -1)$ and another point $(5, 2)$.
The base function is $f(x) = x^2$.
- To find the equation of the parabola, transform f so that the two points are on the graph of the transformed function.
You know that $(2, 4)$ lies on f . Point $(2, 4)$ is transformed into $(5, 2)$.
Determine the necessary transformations and the quadratic equation of the transformed function.
 - $(1, 1)$ is another point on f and it is the point that is transformed into $(5, 2)$. Determine the necessary transformations and the required quadratic equation.
 - Compare your answers from (a) and (b). Explain.



The Chapter Problem—Cryptography

Apply what you have learned about transformations to answer these questions about the Chapter Problem on page 218.

CP15. You could write the code in question CP14 in function notation as follows:

$$f(x) = \text{mod}(17x, 26) + 1, x \in \{1, 2, 3, \dots, 26\}$$

where $\text{mod}(a, b)$ is the remainder when a is divided by b .

For example,

$$\text{mod}(17, 5) = 2 \text{ because } 17 \text{ divided by } 5 \text{ is } 3 \text{ with remainder } 2.$$

- Suppose that the equation is $y = 17x + 1$. What is the equation for the inverse function (the decoder)?
- The last operation is “add 1.” What do you know about $f^{-1}(x)$?
- What is the purpose of taking the remainder after dividing by 26?
- Suppose you multiply a certain number by 17 and divide the result by 26. The remainder is 1. What is the number?
- $f^{-1}(x)$ is very much like $f(x)$, but involves the number you found in (d). Determine $f^{-1}(x)$.
- Verify your answer for all possible values of x .
- Use $f^{-1}(x)$ to decode the message in the Chapter Problem on page 218.