

Extending Algebra Skills: Completing the Square

The same relationship can be expressed in several different algebraic forms. You have seen that quadratic relations can be written several ways:

$$y = -2x^2 + 16x - 24$$

(standard form)

$$y = -2(x - 6)(x - 2)$$

(factored form)

$$y = -2(x - 4)^2 + 8$$

(vertex form)

Each form provides different information about the relationship and its graph. You must be able to move algebraically from one form to another.

You have already completed the square to express a quadratic function in vertex form; however, when you work with real data, the coefficients in the model are often not integers. You will need to use rational numbers to complete the square in these cases.

Example 1

Put $f(x) = -3x^2 + 13x - 11$ in **vertex form**, $f(x) = a(x - h)^2 + k$, by completing the square. State the coordinates of the vertex.

Solution

Complete the square.

$$f(x) = -3x^2 + 13x - 11$$

$$f(x) = -3\left(x^2 - \frac{13}{3}x\right) - 11$$

$$f(x) = -3\left[x^2 - \frac{13}{3}x + \left(-\frac{13}{6}\right)^2 - \left(-\frac{13}{6}\right)^2\right] - 11$$

$$f(x) = -3\left[x^2 - \frac{13}{3}x + \left(-\frac{13}{6}\right)^2\right] - (-3)\left(-\frac{13}{6}\right)^2 - 11$$

$$f(x) = -3\left(x - \frac{13}{6}\right)^2 + \frac{169}{12} - 11$$

$$f(x) = -3\left(x - \frac{13}{6}\right)^2 + \frac{37}{12}$$

The vertex form of $f(x) = -3x^2 + 13x - 11$ is $f(x) = -3\left(x - \frac{13}{6}\right)^2 + \frac{37}{12}$.

The vertex is $\left(\frac{13}{6}, \frac{37}{12}\right)$.

Factor the coefficient of x^2 from the first two terms.

Add and subtract the square of half the coefficient of x inside the brackets.

The first three terms inside the square brackets form a perfect square. Multiply the fourth term by the coefficient of x^2 . Then move this term outside the brackets.

Factor the perfect square.

Simplify.

Example 2

Find the coordinates of the vertex for the graph of $f(x) = -0.16x^2 + 9.76x - 9.408$ by completing the square.

Solution

$$f(x) = -0.16x^2 + 9.76x - 9.408$$

Factor the coefficient of x^2 from the first two terms.

$$f(x) = -0.16(x^2 - 61x) - 9.408$$

Add and subtract the square of half the coefficient of x inside the brackets.

$$f(x) = -0.16[x^2 - 61x + (-30.5)^2 - (-30.5)^2] - 9.408$$

Group the three terms that form the perfect square. Then multiply the fourth term by -0.16 and move it outside the bracket.

$$f(x) = -0.16[x^2 - 61x + (-30.5)^2 - (-0.16)(-30.5)^2 - 9.408]$$

Factor.

$$f(x) = -0.16(x - 30.5)^2 + 148.84 - 9.408$$

Simplify.

$$f(x) = -0.16(x - 30.5)^2 + 139.432$$

The coordinates of the vertex are $(30.5, 139.432)$.

Focus 4.1

Key Ideas

- The same relationship can be expressed in several different algebraic forms.
- Transform the equation of a quadratic function in standard form, $f(x) = ax^2 + bx + c$, to vertex form, $f(x) = a(x - h)^2 + k$, by **completing the square**.

$$f(x) = ax^2 + bx + c$$

Factor the coefficient of x^2 from the first two terms.

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

Add and subtract the square of half the coefficient of x inside the brackets.

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c$$

Group the three terms that form the perfect square. Multiply the fourth term by a and move it outside the brackets.

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] - a\left(\frac{b}{2a}\right)^2 + c$$

Factor the perfect square and simplify.

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Practise, Apply, Solve 4.1

A

1. Factor the coefficient of x^2 from each expression.

(a) $2x^2 - 6x$

(b) $3x^2 + 2x$

(c) $-4x^2 + 2x$

(d) $\frac{1}{2}x^2 - 5x$

(e) $-1.2x^2 - 6.6x$

(f) $\frac{2}{3}x^2 + \frac{5}{6}x$

(g) $-\frac{4}{3}x^2 - 2x$

(h) $2.7x^2 - 0.9x$

(i) $-6x^2 + \frac{3}{5}x$

2. Find the value of d that makes each expression a perfect square.

(a) $x^2 + 3x + d$

(b) $x^2 - 7x + d$

(c) $x^2 + \frac{2}{3}x + d$

(d) $x^2 - \frac{3}{5}x + d$

(e) $x^2 - 2.5x + d$

(f) $x^2 + 0.45x + d$

(g) $x^2 - 22.4x + d$

(h) $x^2 - \frac{5}{9}x + d$

(i) $x^2 + \frac{x}{100} + d$

3. Express each function in vertex form by completing the square.

(a) $f(x) = x^2 - 4x + 3$

(b) $f(x) = x^2 + 6x - 8$

(c) $f(x) = 3x^2 - 6x + 1$

(d) $f(x) = x^2 - x$

(e) $f(x) = x^2 + 3x + 5$

(f) $f(x) = -x^2 + 5x - 2$

(g) $f(x) = x^2 - 2.6x - 2.7$

(h) $f(x) = x^2 + 5.4x - 1.8$

(i) $f(x) = x^2 - 3.4x - 2.8$

4. Express $g(x) = 2x^2 + 5x + 4$ in vertex form. Explain your steps.

B

5. Express each equation in vertex form by completing the square.

(a) $f(x) = x^2 - 3x + 2$

(b) $f(x) = 2x^2 + 5x - 4$

(c) $f(x) = 3x^2 - 4x + 1$

(d) $f(x) = 5x^2 - x - 5$

(e) $f(x) = -2x^2 + 3x + 5$

(f) $f(x) = -4x^2 - 7x - 2$

(g) $f(x) = 1.4x^2 - 4.9x - 2.7$

(h) $f(x) = -3.6x^2 + 5.4x - 1.67$

(i) $f(x) = \frac{1}{2}x^2 - 4x + 1$

(j) $f(x) = \frac{2}{3}x^2 - 5x - 2$

(k) $f(x) = -\frac{3}{2}x^2 - \frac{7}{8}x + \frac{1}{2}$

(l) $f(x) = -\frac{4}{5}x^2 + \frac{2}{3}x - \frac{3}{10}$

6. i. Find the coordinates of the vertex by completing the square.

ii. State the domain and the range of $f(x)$.

iii. Sketch the graph for each quadratic function.

(a) $f(x) = x^2 + 10x + 24$

(b) $f(x) = 4x^2 + 8x + 7$

(c) $f(x) = -2x^2 + 5x - \frac{1}{8}$

(d) $f(x) = \frac{1}{2}x^2 - 3x + 4$

(e) $f(x) = -2x^2 + 7x - 12$

(f) $f(x) = -2x^2 + 7x$

7. The function $d(x) = 5x^2 - 3x + 1$ is quadratic.
- Write the equation in vertex form.
 - Write the equation of the axis of symmetry.
 - Write the coordinates of the vertex.
 - What is the maximum or minimum value of d ?
 - What is the domain of d ?
 - What is the range of d ?
 - Sketch the graph of d .
8. A football is punted into the air. Its height h , in metres, after t seconds is given by $h(t) = 1 + 24.5t - 4.9t^2$.
- Find the maximum height of the ball by completing the square. Round your answer to the nearest hundredth of a metre.
 - When does the ball reach its maximum height?
9. For each function, determine the coordinates of the vertex.
- $f(x) = 2x^2 - \frac{5}{6}x - \frac{1}{2}$
 - $f(x) = -3x^2 - \frac{2}{3}x + \frac{3}{4}$
 - $f(x) = \frac{1}{2}x^2 + 4x - 8$
 - $f(x) = -6x^2 + \frac{6}{7}x - \frac{3}{7}$
10. What transformations must you apply to the graph of $y = x^2$ to obtain the graph of $y = 0.4x^2 - 2x + 3.5$? Show all your work and explain each step.
11. Researchers found that the relationship between the yield of a crop and the amount of watering could be modelled by $y = -1478 + 3345x - 446x^2$, where y is the yield, in kilograms per hectare (kg/ha), and x is the annual amount of water, in hectare-metres (ha-m). How much water produces the greatest yield?



12. What numbers could t be if t^2 must be less than $12t - 20$?
13. State the domain and range.
- $g(x) = x^2 + 3x - 4$
 - $h(x) = -2x^2 + 5x - 1$
 - $f(x) = \frac{1}{2}x^2 - \frac{2}{3}x + \frac{3}{4}$
 - $k(x) = 0.1x^2 - 0.5x + 0.3$
14. **Check Your Understanding:** Why should you be able to write the equation of the same quadratic function in several different equivalent forms? What are the steps for changing the equation of a quadratic function from standard form into vertex form? Use an example to explain.