

Part 1: Familiar Number Systems

The figures on this clay tablet represent numbers that an ancient Sumerian used to record sheep and goats. The numbers that are commonly used to calculate, count, and measure have evolved over time to meet the needs of people with new kinds of problems to be solved. In the following problems, note how the numbers change.



Think, Do, Discuss

- Lia has some CDs and buys four more. She now has 28 CDs. How many CDs did she have originally? Write an equation. Let the variable be the original number of CDs. What is the solution? What kind of numbers did you use in your solution?
- Craig builds seven lawn chairs in two days. How many chairs does he build in a day? Write an equation. Let the variable be the original number of chairs that Craig builds in a day. Solve your equation. What kind of numbers did you use in your solution?
- Stuart buys nine CDs. He then has a total of nine CDs. How many CDs did he have originally? Write an equation. Let the variable be the original number of CDs. What is the solution? This number was not introduced into use until the tenth century!
- The temperature rose 10°C and is now 6°C . What was the initial temperature? Write an equation. Let the initial temperature be the variable. Solve the equation. What kind of numbers did you use in your solution?
- Kim wants to create a square enclosure for her vegetable garden with an area of 2 m^2 . How much chicken wire will she need for fencing the garden? Write an equation. Let the variable be the length of each side of the square enclosure. Solve the equation. What type of number did you use in your solution?
- The above questions refer to the following sets of numbers:
 - the set of natural numbers, $\mathbf{N}$, $\{1, 2, 3, 4, \dots\}$
 - the set of whole numbers, $\mathbf{W}$, $\{0, 1, 2, 3, 4, \dots\}$
 - the set of integers, $\mathbf{I}$, $\{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$
 - the set of rational numbers, $\mathbf{Q}$, $\{\frac{a}{b}: a, b \in \mathbf{I}, b \neq 0\}$
 - the set of irrational numbers, $\tilde{\mathbf{Q}}$. An irrational number cannot be expressed as the ratio of two integers, for example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\dots$.

Are these sets distinct, or do some numbers belong to more than one set? Give examples to illustrate your answer. Draw a diagram to show how the sets connect with each other.

Part 2: Defining the Set of Complex Numbers

The sets described in Part 1 are members of the set of **real numbers**. The set of real numbers, $\mathbf{R}$, has both rational and irrational numbers.

A quadratic function whose graph does not cross or touch the x -axis has no zeros. Take for example, the function $f(x) = 0.5x^2 - 4x + 10$ and the corresponding quadratic equation $0 = 0.5x^2 - 4x + 10$. In the set of real numbers, this equation has no real roots. However, if we extend the number system, you can find roots for this and other equations.

Think, Do, Discuss

1. Sketch the graph of the function $f(x) = x^2 + 2x + 2$. How many zeros does $f(x)$ have?
2. Try to find the zeros of the function $f(x) = x^2 + 2x + 2$ by letting $f(x) = 0$ and using the quadratic formula.
3. Why are you unable to find the zeros?
4. Let a non-real number be i , such that $i^2 = -1$. Then you can write any negative number as a multiple of i^2 . For example, you could write -4 as $4i^2$ and write -31 as $31i^2$, and so on. Now you can solve the equation $x^2 + 4 = 0$.

$$\begin{aligned}x^2 + 4 &= 0 \\x^2 &= -4 \\x^2 &= 4i^2 \\x &= \pm 2i\end{aligned}$$

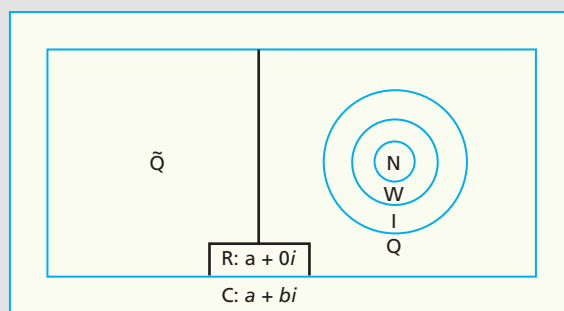
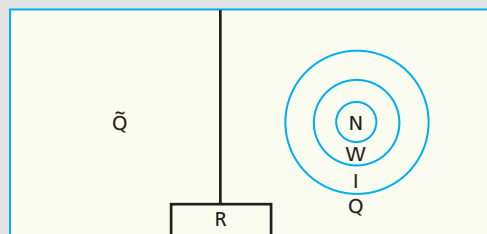
What are the solutions of $x^2 + 25 = 0$? of $x^2 + 3 = 0$?

5. The number i has been called an **imaginary number**. This name is easy to remember, but it is misleading because the number i does exist. Some quadratic equations have solutions that have real and imaginary parts. Try to solve the equation $x^2 + 2x + 2 = 0$. Use the number i to help you obtain two roots.
6. The roots in step 5 should be in the form $a + bi$ and $a - bi$. These numbers are called **complex numbers**. Notice that complex numbers have a real part, a , and an imaginary part, $\pm bi$. Would you describe real numbers as complex? Explain your answer.

Focus 4.4

Key Ideas

- We use the set of real numbers, R , most frequently to describe the relationships in the world around us. The set of real numbers contains all rational and irrational numbers. It also contains the sets of natural numbers, whole numbers, and integers.
- You can write any complex number in the form $a + bi$, where a and b are real numbers and i is a non-real number such that $i^2 = -1$.
- Since you can write any real number in the form $a + 0i$, then the complex numbers, C , include all the real numbers.
- Using complex numbers, you can find the square root of any real number—positive, zero, or negative.
- A quadratic equation, with no real roots, has two complex roots in the form $a + bi$ and $a - bi$. These roots form a **conjugate pair**. If $b^2 - 4ac < 0$, then a quadratic equation has two complex conjugate roots.



Example 1

Solve for x , where x is a complex number. Round to two decimal places, if necessary.

(a) $x^2 + 36 = 0$

(b) $2x^2 + 50 = 0$

(c) $5x^2 + 60 = 0$

Solution

(a) $x^2 + 36 = 0$

$$x^2 = -36$$

$$x^2 = 36(-1)$$

$$x^2 = 36i^2$$

$$x = \pm\sqrt{36i^2}$$

$$x = \pm 6i$$

(b) $2x^2 + 50 = 0$

$$2x^2 = -50$$

$$x^2 = -25$$

$$x^2 = 25i^2$$

$$x = \pm\sqrt{25i^2}$$

$$x = \pm 5i$$

(c) $5x^2 + 60 = 0$

$$5x^2 = -60$$

$$x^2 = -12$$

$$x^2 = 12i^2$$

$$x = \pm\sqrt{12i^2}$$

$$x = \pm 3.46i$$

Example 2

Find the complex roots of the equation $4x^2 - 2x + 3 = 0$.

Solution

Substituting $a = 4$, $b = -2$, and $c = 3$ in the quadratic formula gives

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(3)}}{2(4)} \\x &= \frac{2 \pm \sqrt{4 - 48}}{8} \\x &= \frac{2 \pm \sqrt{-44}}{8} \\x &= \frac{2 \pm \sqrt{44}i^2}{8} \\x &= \frac{2 \pm 6.63i}{8}\end{aligned}$$

The complex roots of the equation are $x = 0.25 + 0.83i$ and $x = 0.25 - 0.83i$.

Example 3

One root of a quadratic equation is $2 + 5i$. Find the other root and express the quadratic equation in factored form.

Solution

If a quadratic equation has complex roots, then the roots form a conjugate pair. If one root is $2 + 5i$, then the other root is $2 - 5i$. The quadratic equation with roots $2 + 5i$ and $2 - 5i$ in factored form is

$$[x - (2 + 5i)] \times [x - (2 - 5i)] = 0, \text{ or } (x - 2 - 5i)(x - 2 + 5i) = 0$$

Practise, Apply, Solve 4.4

A

1. For each number, identify the set(s) of numbers to which it belongs.

(a) -3

(b) $\frac{3}{4}$

(c) $5 + 3i$

(d) $\sqrt{7}$

(e) 6.35

(f) 5.23

(g) $-\frac{2}{3}$

(h) 1

2. Write each of the following as a complex number. Round to two decimal places, where necessary.

(a) $\sqrt{-16}$

(b) $\sqrt{-121}$

(c) $-\sqrt{-9}$

(d) $\sqrt{-18}$

(e) $-\sqrt{-28}$

(f) $\pm\sqrt{-50}$

3. Write the complex conjugate of each complex number.
- (a) $1 + i$ (b) $2 - 3i$ (c) $-3 - 4i$
 (d) $-7 + 5i$ (e) $11 - 11i$ (f) $2 + \sqrt{2}i$
4. Determine whether the given complex number is a solution to the equation.
- (a) $x^2 = -16, 4i$ (b) $x^2 = -14, 7i$
 (c) $x^2 + 6 = 5, i$ (d) $x^2 + 10 = 16, 3i$
5. Solve for x , where x is a complex number. Round to two decimal places, if necessary.
- (a) $x^2 + 4 = 0$ (b) $x^2 + 64 = 0$ (c) $3x^2 + 27 = 0$
 (d) $-2x^2 = 98$ (e) $x^2 + 32 = 0$ (f) $-3x^2 - 150 = 0$
6. Solve each quadratic equation, where x is a complex number. Round to two decimal places, if necessary.
- (a) $x^2 - 4x + 5 = 0$ (b) $x^2 - 2x + 17 = 0$
 (c) $-x^2 + 4x = 13$ (d) $2x^2 + 2x + 1 = 0$
 (e) $2x^2 = x - 1$ (f) $3x(x + 2) = -18$
- B**
7. For each quadratic equation, in the form $ax^2 + bx + c = 0$, determine the value of the discriminant, $b^2 - 4ac$. What does this value tell you about each equation?
- (a) $x^2 + 2x + 5 = 0$ (b) $x^2 - 2x - 3 = 0$
 (c) $5x^2 - 4x + 2 = 0$ (d) $3x^2 - 24x + 48 = 0$
8. Solve each equation in question 7. Round to two decimal places, if necessary.
9. **Knowledge and Understanding:** Find the roots, real or complex, of each quadratic equation, using an appropriate method. Express all answers to two decimal places.
- (a) $x^2 + 5x - 24 = 0$ (b) $3(x - 2)^2 - 15 = 0$
 (c) $3x^2 - 6x + 1 = 0$ (d) $x^2 - 2x + 5 = 0$
10. In each of the following, one root of a quadratic equation is given. State the other root and write the equation in factored form.
- (a) $2i$ (b) $2 - i$ (c) $1 + 3i$
 (d) $-1 - i$ (e) $3 - 4i$ (f) $5i - 12$
11. Find the complex factors of each expression.
- (a) $x^2 + 4x + 5$ (b) $x^2 + 4x + 13$
 (c) $x^2 - x + 1$ (d) $x^2 - 2x + 26$
12. Solve for x , where x is a complex number. Express answers to two decimal places.
- (a) $3x^2 + 2x + 1 = 0$ (b) $2x^2 - 3x + 7 = 0$
 (c) $3x^2 - 2x + 10 = 0$ (d) $5x^2 + x + 7 = 0$

13. Find two numbers with a sum of 4 and a product of 13.
14. Find two numbers with a sum of 5 and a product of 7.
15. **Application:** The monthly sales, n , in thousands of units, of a new perfume can be modelled by the function $n(t) = -0.75t^2 + 6t + 51$, where $t = 0$ corresponds to the first month of sales. Can the company expect to sell 65 000 units in any one month over the next year? Justify your answer graphically and algebraically.
16. **Communication:** Explain how complex numbers are different from real numbers. Describe the real number system and use examples to illustrate your explanation.
17. **Thinking, Inquiry, Problem Solving:** Investigate, using a graphing calculator, adding, subtracting, and multiplying complex numbers. You will need to change the real mode setting (**MODE**) from **real** to **a+bi**. Is the sum of two complex numbers a complex number? Is the product of two complex numbers always a complex number? Give examples to justify your answers.
18. **Check Your Understanding:** Explain what complex numbers are and how they help in the solution of quadratic equations. Use examples to illustrate your explanations.



19. Find all the roots, real and complex, of the equation $x^4 - 81 = 0$.
20. Is it possible to put complex numbers in order? If so, then how would you order complex numbers? If not, then why not?
21. Verify that $(x + 1)(x^2 - x + 1) = x^3 + 1$. Find all the roots, real and complex, of the equation $x^3 + 1 = 0$.



The Chapter Problem—Fundraising

In this section, you found complex roots for quadratic equations that did not have real roots. Apply what you have learned to answer these questions about the Chapter Problem on page 300.

- CP10.** Examine each profit function you obtained in CP6. Will the student council be able to raise \$7000 by selling T-shirts?
- CP11.** Determine the maximum amount the student council could raise on the basis of each supplier.
- CP12.** Find the number of T-shirts sold in each case in CP11.
- CP13.** Which T-shirt supplier would you choose? Explain your choice.