

Adding, Subtracting, and Multiplying Complex Numbers

Because complex numbers have two parts, the real part and the imaginary part, they are ideal for describing real-life quantities that have two components. For example, in electronics, the resistance of a circuit is expressed as impedance. Impedance has two parts, resistance, R , and a reactance, X . Impedance is represented by the complex number, $R + Xi$.

In this section, you will learn how to add, subtract, and multiply complex numbers. You may be interested to know that a calculator uses complex arithmetic algorithms to perform many simple operations.

Adding and Subtracting Complex Numbers

When adding or subtracting two complex numbers, use the laws of algebra and treat the symbol i as if it were a variable.

Example 1

- (a) Add $(5 - 2i)$ and $(-3 + 7i)$. (b) Find the sum of $(-4 - 3i)$ and its complex conjugate.

Solution

(a) $(5 - 2i) + (-3 + 7i)$

$$\begin{array}{l}
 \text{real} \swarrow \quad \searrow \\
 = 5 - 2i - 3 + 7i \\
 \quad \quad \quad \nearrow \quad \nwarrow \\
 \text{imaginary} \\
 = 5 - 3 - 2i + 7i \\
 = 2 + 5i
 \end{array}$$

(b) $(-4 - 3i) + (-4 + 3i)$

$$\begin{aligned}
 &= -4 - 3i - 4 + 3i \\
 &= -4 - 4 - 3i + 3i \\
 &= -8 + 0i
 \end{aligned}$$

The sum is a real number.

Example 2

- (a) Simplify $(7 - 4i) - (2 + 3i)$. (b) Subtract $(3 - 5i)$ from its complex conjugate.

Solution

(a) $(7 - 4i) - (2 + 3i)$

$$\begin{aligned}
 &= 7 - 4i - 2 - 3i \\
 &= 7 - 2 - 4i - 3i \\
 &= 5 - 7i
 \end{aligned}$$

(b) $(3 + 5i) - (3 - 5i)$

$$\begin{aligned}
 &= 3 + 5i - 3 + 5i \\
 &= 3 - 3 + 5i + 5i \\
 &= 0 + 10i \\
 &= 10i
 \end{aligned}$$

The difference is an imaginary number.

Multiplying Complex Numbers

Use the distributive law to multiply complex numbers.

Example 3

(a) Simplify.

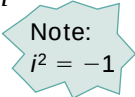
i. i^2 ii. $(i^2)^5$ iii. i^7

(b) Multiply $(4 + 3i)$ by $(2 - i)$.

(c) Expand and simplify $(-3 - 2i)^2$.

Solution

(a) i. By definition, $i^2 = -1$. ii. $(i^2)^5 = (-1)^5 = -1$ iii. $i^7 = i^6 \times i$
 $= (i^2)^3 \times i$
 $= (-1)^3 \times i$
 $= -i$

(b) $(4 + 3i)(2 - i)$
 $= 8 - 4i + 6i - 3i^2$
 $= 8 + 2i - 3(-1)$ 
 $= 8 + 3 + 2i$
 $= 11 + 2i$

(c) $(-3 - 2i)^2$
 $= (-3 - 2i)(-3 - 2i)$
 $= 9 + 6i + 6i + 4i^2$
 $= 9 + 12i + 4(-1)$
 $= 9 - 4 + 12i$
 $= 5 + 12i$

Multiplying Complex Conjugates

The letter z is often used to denote a complex number, and $\bar{z}$ is the complex conjugate of z . If $z = a + bi$, then $\bar{z} = a - bi$.

Example 4

(a) Find the product of $(5 - 2i)$ and its complex conjugate.

(b) If $z = a + bi$, then find $z\bar{z}$.

Solution

(a) $(5 - 2i)(5 + 2i)$
 $= 25 + 10i - 10i - 4i^2$
 $= 25 + 0i - 4(-1)$
 $= 29 + 0i$
 $= 29$

(b) $(a + bi)(a - bi)$
 $= a^2 - abi + abi - b^2i^2$
 $= a^2 + 0i - b^2(-1)$
 $= a^2 + b^2 + 0i$
 $= a^2 + b^2$

Notice that the result of multiplying two complex conjugates is always a real number.

Key Ideas

- To add or subtract complex numbers, combine the real terms and the imaginary terms separately.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

- To multiply two complex numbers, use the distributive law.

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

- If $z = a + bi$ is a complex number, then $\bar{z}$ is the complex conjugate of z , and $\bar{z} = a - bi$.
- The product of two complex conjugates is always a real number, since $z\bar{z} = a^2 + b^2$.
- Two complex numbers are equal if, and only if, their real parts are equal and their imaginary parts are equal. That is, $a + bi = c + di$ if, and only if, $a = c$ and $b = d$.
- A complex number is zero if, and only if, the real term and the imaginary term are both 0. That is, $a + bi = 0$ if, and only if, $a = 0$ and $b = 0$.

Practise, Apply, Solve 4.5U**A**

- Add the pairs of complex numbers.

(a) $2 + 6i$ and $5 - i$

(b) $4 - 3i$ and $1 + i$

(c) $3 + 7i$ and $-4 - 9i$

(d) $-1 + 3i$ and $2 - 13i$

(e) $5i$ and $1 - 2i$

(f) $3 + \sqrt{2}i$ and $\sqrt{2}i$

- Subtract the second number from the first in each part of question 1.

- Simplify each of the following.

(a) $(2 + i)(3 + 5i)$

(b) $(5 - 3i)(7 + i)$

(c) $(4 - 3i)(-2 - i)$

(d) $i(5 + i)$

(e) $(2 + i)^2$

(f) $(2 + 3i)(2 - 3i)$

- Find $z\bar{z}$ for each value of z .

(a) $z = 1 + i$

(b) $z = 3 - 2i$

(c) $z = -4 + 2i$

5. Simplify.

(a) $(3 + 4i) + (2 + 3i)$

(b) $(5 + i) - (3 - 2i)$

(c) $(2i)^2$

(d) i^4

(e) $\frac{1}{i^2}$

(f) $\frac{1}{(1+i)(1-i)}$

(g) $(1 + 2i)^2$

(h) $i(4 - 5i)^2$

(i) $(2 - i)(1 + 2i)(1 - i)$

(j) $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2$

(k) $(6 - 4i)^2 - (1 - 3i)^2 + 5i$

(l) $\sqrt{-4} + \sqrt{-25}$

B

6. For each pair of roots, find a quadratic equation. Express your equation in the form $ax^2 + bx + c = 0$.

(a) $2i, -2i$

(b) $1 + 3i, 1 - 3i$

(c) $3 + 4i, 3 - 4i$

(d) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

7. One root of a quadratic equation is $3 - i$. Determine the other root and write a quadratic equation that has these roots. Express your answer in the form $ax^2 + bx + c = 0$.

8. Show by substitution that $x = 2 + i$ is a solution of the equation $x^2 - 4x + 5 = 0$, where x is a complex number.

9. If $f(x) = 2x^2 + 3x - 2$, then find $f(2 - i)$.

10. Find $g(3 - 4i)$ if $g(x) = -3x^2 - 2x + 4$.

11. Write a quadratic equation that has complex solutions of the form $a + bi$, where $a \neq 0$ and $b \neq 0$. Solve the equation. Show by substitution that the solutions you found are correct.

12. Suppose that $a + bi = 0$ and that $a \neq 0$ and $b \neq 0$. Rearrange the equation $a + bi = 0$ to isolate i . Explain why this rearrangement contradicts what you know about i .

13. If $z = a + bi$, then find the values of a and b for which $z + \bar{z} = 10$ and $z - \bar{z} = 14i$.

14. Check Your Understanding: If $z = -2 + 3i$ and $w = 4 - i$, then find $z + w$, $z - w$, zw , w^2 , and $z\bar{z}$.

C

15. Find the square roots of the complex number $5 + 12i$.

16. If $a + bi$ is a root of the quadratic equation $x^2 + cx + d = 0$, then show that $a^2 + b^2 = d$ and $2a + c = 0$.