



Part 1: Comparing Hourly Rates

Stuart is trying to decide which of two job offers to accept for summer employment. He could work as a shipper at the local lumber store, where he would be paid \$11.50/h. The other job is with a student-operated house-painting business where he would earn \$450 for each house that is painted.



Think, Do, Discuss

1. What information does Stuart need to calculate the hourly rate for painting houses?
2. If Stuart takes 40 h to paint a house, what is the hourly rate?
3. Suppose Stuart wants to determine the hourly rate of pay if he takes less than 40 h to paint a house, or if he takes more than 40 h to paint a house. Copy and complete the table to show the hourly rate, y , for the hours worked, x , to paint one house.

Hours Worked, x (h)	1	10	15	20	25	30	35	40	45	50	55
Hourly Rate, y (\$/h)											

4. What happens to the hourly rate as the number of hours increases? Explain why the hourly rate is represented by y and why the number of hours worked is represented by x .
5. Write an equation for the relationship between the hourly rate, y , and the number of hours worked, x . Enter the equation into a graphing calculator and graph it. Restrict the domain to only positive x -values.
6. Describe the graph. According to the graph, what happens to y as x becomes very small, that is, close to 0? What happens to y as x becomes very large?
7. How would the graph change if the rate of pay increased to \$550 per house?
8. For each line in the table, calculate the product xy . What do you notice? Why does this happen? This type of relationship is called inverse variation. What other quantities vary in this way?
9. What factors, other than the speed at which Stuart can paint, might influence the time it takes to paint a house? Which job would you advise Stuart to take? Explain your answer.

Part 2: The Reciprocal Function $f(x) = \frac{1}{x}$

In Part 1, the value of one variable in the relation increased while the value of the other variable decreased. The products of corresponding x - and y -values were constant. This relation can be described by $xy = k$, or $y = \frac{k}{x}$, and is called **inverse variation**. There are many real-life quantities that vary in this way. They can be modelled by functions in the form $f(x) = \frac{k}{x}$. The simplest of these functions is $f(x) = \frac{1}{x}$.

Think, Do, Discuss

1. $f(x) = \frac{1}{x}$ is a reciprocal function. Explain why this name is appropriate.
2. Enter the function into a graphing calculator and adjust the window (Xmin=-5, Xmax=5, Ymin=-5, and Ymax=5). Then graph. Describe the shape of the graph and any special features. This graph is called a hyperbola.

Use the Table feature of the graphing calculator to do the next few questions.

3. What happens to the value of y as x becomes very large, for example, greater than 1000? What happens to y as x becomes very small, for example, less than -1000? Describe the graph for both situations. Does the graph meet or cross the x -axis? How do you know?
4. What is true about the value of y when (a) x is close to 1? (b) x is close to -1?
5. What happens to the value of y as x decreases from 1 to 0? as x increases from -1 to 0?
6. In the table, what is the y -value for $x = 0$? What does this y -value mean? What is the domain for the function $f(x) = \frac{1}{x}$? What is the range?
7. As x gets closer and closer to 0, the graph approaches the y -axis but never meets it. The y -axis is an example of a **vertical asymptote**. The graph also has a **horizontal asymptote**. What is its equation? How do the equations of the asymptotes relate to the domain and the range of f ? Since the asymptotes are perpendicular for this hyperbola, it is called a **rectangular hyperbola**.
8. Find the equation for the inverse of $f(x) = \frac{1}{x}$, that is, $f^{-1}(x)$. What do you notice? Does this result seem reasonable from the graph? Explain your answer.
9. The inverse of a function is written as $f^{-1}(x)$. However, $[f(x)]^{-1}$ refers to the reciprocal function. Explain the difference between the inverse of a function, $f^{-1}(x)$, and the reciprocal of a function, $\frac{1}{f(x)}$. Use examples of each function and discuss the domain and the range in each case.
10. Using your knowledge of transformations, how would the graph of each function compare to the graph of $f(x) = \frac{1}{x}$? Check using a graphing calculator.
(a) $f(x) = \frac{5}{x}$ [Hint: $\frac{5}{x} = 5\left(\frac{1}{x}\right)$] (b) $f(x) = \frac{100}{x}$ (c) $f(x) = \frac{-1}{x}$
11. Write each function in question 10 as a power of x . In this form they are *power functions*.

Part 3: Making Connections Between the Graphs of Functions and Their Reciprocals

Can you tell from the graph of a function how the graph of the reciprocal of that function will look? In this investigation, you will compare the graph of $y = f(x)$ with the graph of $y = \frac{1}{f(x)}$ to discover some common properties.

Think, Do, Discuss

1. On a graphing calculator, adjust the window to (Xmin=-5, Xmax=5, Ymin=-5, and Ymax=5).

Graph $y = x$ and $y = \frac{1}{x}$ on the same axes. At what points do the graphs intersect? What are the y-coordinates of the points of intersection? Explain why these values make sense.

2. Where is the zero for the function $y = x$? What feature of the graph of $y = \frac{1}{f(x)}$ exists at this x-value? Explain why this makes sense.
3. What do you notice about the quadrants in which each graph occurs? When $y = x$ is close to the x-axis, where is $y = \frac{1}{f(x)}$? When $y = x$ is far away from the x-axis, where is $y = \frac{1}{f(x)}$?

4. Repeat steps 1 to 3 for each pair of functions. When you enter the reciprocal functions into a graphing calculator, you will need to include brackets around each denominator. For example, enter the function $y = \frac{1}{x+1}$ as $Y2=1/(X+1)$.

(a) $y = x + 1$ and $y = \frac{1}{x+1}$

(b) $y = x - 2$ and $y = \frac{1}{x-2}$

(c) $y = 2x + 5$ and $y = \frac{1}{2x+5}$

(d) $y = 3x - 4$ and $y = \frac{1}{3x-4}$

5. On paper, sketch the graph of $y = 2x - 1$. Using your answers for steps 1 to 4 as a guide, sketch the graph of $y = \frac{1}{2x-1}$. Use a calculator to verify your sketch.

6. On a graphing calculator, change the window to (Xmin=-4.7, Xmax=4.7, Ymin=-5, and Ymax=5). Graph the functions $y = (x - 3)(x + 1)$ and $y = \frac{1}{(x - 3)(x + 1)}$ on the same axes. You will need an extra pair of brackets around the product in the denominator of the reciprocal function: $Y2=1/((X-3)(X+1))$.

- (a) At what points do the graphs intersect?
- (b) What features of the graph of the reciprocal function exist at the zeros of the quadratic function?
- (c) Compare the shape of the graph of the reciprocal function with the shape of the parabola. Consider the lines of symmetry, the quadrants that each graph occupies, and the position of each graph.

7. Repeat step 6 for each of the following pairs of functions.

(a) $y = (x + 2)(x - 1)$, $y = \frac{1}{(x + 2)(x - 1)}$

(b) $y = (1 - x)(4 + x)$, $y = \frac{1}{(1 - x)(4 + x)}$

(c) $y = (x - 2)^2$, $y = \frac{1}{(x - 2)^2}$

8. Sketch the parabola for the function $y = x^2 + x - 6$. First factor. Where are the vertical asymptotes of the graph of $y = \frac{1}{x^2 + x - 6}$? Draw the asymptotes.

9. What points will the two graphs in question 8 have in common? Mark these points on the sketch. For the graph above the x -axis, decide what will happen to the graph of the reciprocal function to the right and left of the common points. Use the distance of the parabola from the x -axis as a guide. Sketch those parts of the graph for $y = \frac{1}{x^2 + x - 6}$.

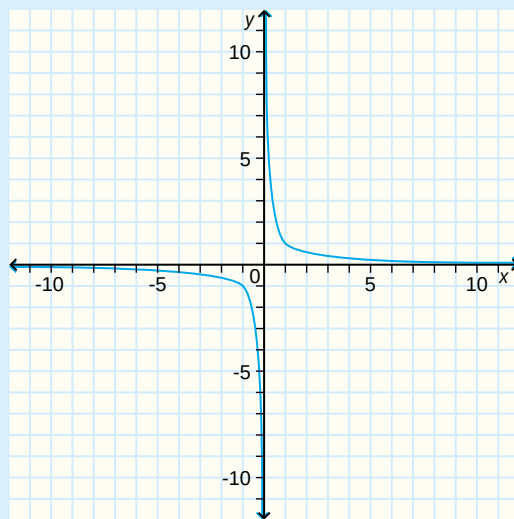
10. Determine the coordinates of the vertex. Use the y -coordinate of the vertex to find the value of $\frac{1}{x^2 + x - 6}$ corresponding to this y -value. Mark this point on the sketch. Then decide what will happen to the graph of the reciprocal function on each side of the common points *below* the x -axis. Sketch this final part of the graph for $y = \frac{1}{x^2 + x - 6}$. Use a graphing calculator to verify your sketch.

Focus 4.6

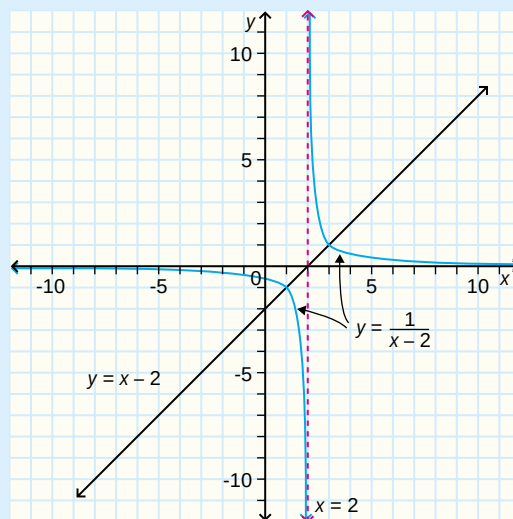
Key Ideas

- A relation has two variable quantities. If one quantity increases as the other decreases, and the product of the two quantities remains constant, then the quantities vary inversely. The relation displays inverse variation.
- An inverse variation can be described by the equations $xy = k$, $y = \frac{k}{x}$, or $y = kx^{-1}$.
- The simplest function that displays inverse variation is $f(x) = \frac{1}{x}$.
- The graph of $y = \frac{1}{x}$ shown here is called a hyperbola.

This hyperbola has two congruent parts, one in each of the first and third quadrants. In real-world applications, often only one branch is needed.



- The domain of $f(x) = \frac{1}{x}$ is the set of real numbers except 0, $D = \{x \mid x \in \mathbf{R}, x \neq 0\}$. The range is also the set of real numbers except 0, $R = \{y \mid y \in \mathbf{R}, y \neq 0\}$.
- The inverse of $f(x) = \frac{1}{x}$ is $f^{-1}(x) = \frac{1}{x}$.
- An asymptote is a line that a function approaches but does not touch or intersect. The graph of $y = \frac{1}{x}$ has a vertical asymptote with equation $x = 0$ and a horizontal asymptote with equation $y = 0$. In this case, the vertical asymptote occurs at the break in the domain of the function, and the horizontal asymptote occurs at the break in the range. Because these two asymptotes are perpendicular, the hyperbola is called a rectangular hyperbola. In other types of hyperbolas, the asymptotes are not necessarily perpendicular.
- You can deduce the shape of the graph of $y = \frac{1}{f(x)}$ from the graph of $y = f(x)$ using the following properties:
 - ◆ At each zero of $f(x)$, the graph of $y = \frac{1}{f(x)}$ has a vertical asymptote.
 - ◆ The graph of $y = \frac{1}{f(x)}$ always passes through the points where $f(x) = 1$ or $f(x) = -1$.
 - ◆ When $f(x)$ is close to 1, $\frac{1}{f(x)}$ is also close to 1.
When $f(x)$ is close to -1 , $\frac{1}{f(x)}$ is also close to -1 .
 - ◆ Both graphs occur in the same quadrants.
When $f(x)$ is positive, $\frac{1}{f(x)}$ is positive.
When $f(x)$ is negative, $\frac{1}{f(x)}$ is negative.
 - ◆ As $f(x)$ gets larger, $\frac{1}{f(x)}$ gets smaller. As $f(x)$ gets smaller, $\frac{1}{f(x)}$ gets larger.



Example 1

The distance from Ottawa to Sault Ste. Marie is about 850 km. Create a table showing the time, x , in hours, for the journey at an average speed, y , of 30, 40, 50, 60, 70, 80, and 90 km/h. Sketch a graph to show the relationship between x and y . Write an equation for the relation. What is the effect on the time if the average speed is doubled?

Solution

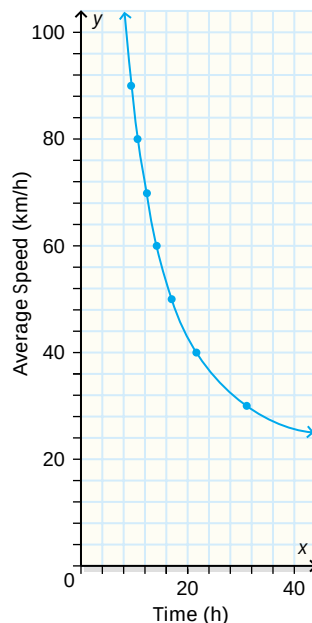
Time, x (h)	Average Speed, y (km/h)
28.3	30
21.2	40
17.0	50
14.2	60
12.1	70
10.6	80
9.4	90

An equation for the relation is $y = \frac{850}{x}$, or $xy = 850$.

In the table, you can see that the time at an average speed of 60 km/h is half the time at 30 km/h, and the time at 80 km/h is half the time at 40 km/h.

$$\begin{aligned} \text{If } y &= \frac{850}{x}, \text{ then } 2y = \frac{850}{x} \times 2. \\ &= \frac{850}{\frac{x}{2}} \end{aligned}$$

The effect of doubling the average speed is to halve the time of the journey.

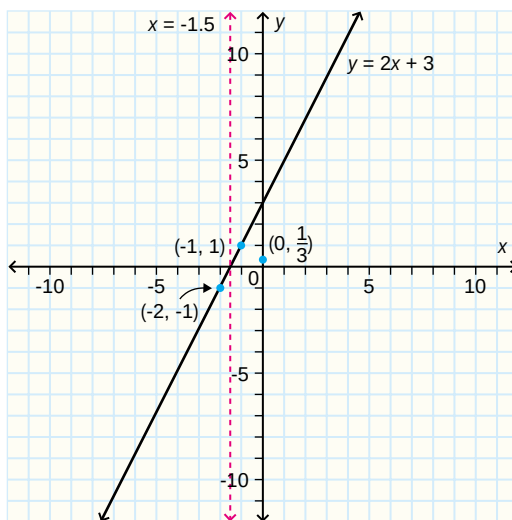


Example 2

Sketch the graph of the function $f(x) = \frac{1}{2x+3}$ by first sketching the graph of $f(x) = 2x + 3$.

Solution

The line $y = 2x + 3$ has slope 2 and y -intercept 3. The x -intercept is -1.5 .
The value of y is 1 when $x = -1$.
The value of y is -1 when $x = -2$.

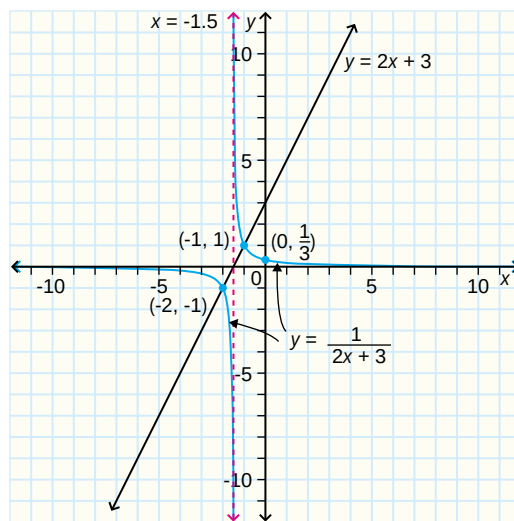


Given this information, you know the graph of $y = \frac{1}{2x+3}$

- has a vertical asymptote with equation $x = -1.5$
- passes through points $(-1, 1)$ and $(-2, -1)$
- has y -intercept $\frac{1}{3}$

Also, given the line $y = 2x + 3$, you know the graph of $y = \frac{1}{2x+3}$

- is in the first quadrant for $x > 0$
- is in the second quadrant for $-1.5 < x < 0$
- is in the third quadrant for $x < -1.5$



Finally the graph of $y = \frac{1}{2x+3}$ will be far from the x -axis when the line $y = 2x + 3$ is close to the x -axis. The graph of $y = \frac{1}{2x+3}$ will be close to the x -axis when the line $y = 2x + 3$ is far from the x -axis.

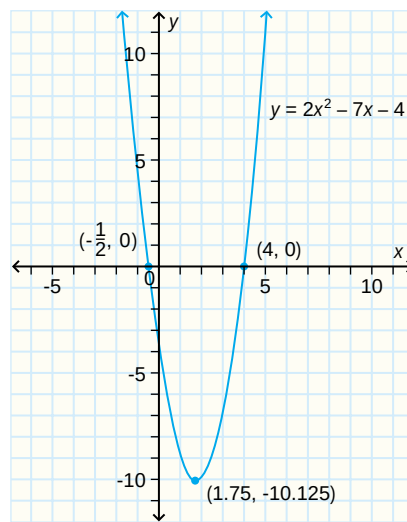
Example 3

Sketch the graph of $y = \frac{1}{2x^2 - 7x - 4}$ by first graphing $y = 2x^2 - 7x - 4$.

Solution

The graph of $y = 2x^2 - 7x - 4$ is a parabola that opens up. From the factored form, $y = (2x + 1)(x - 4)$, you know that the graph crosses the x -axis at $x = -\frac{1}{2}$ and $x = 4$. The x -coordinate of the vertex is $\frac{-0.5 + 4}{2} = 1.75$.

The y -coordinate of the vertex is $2(1.75)^2 - 7(1.75) - 4 = -10.125$.



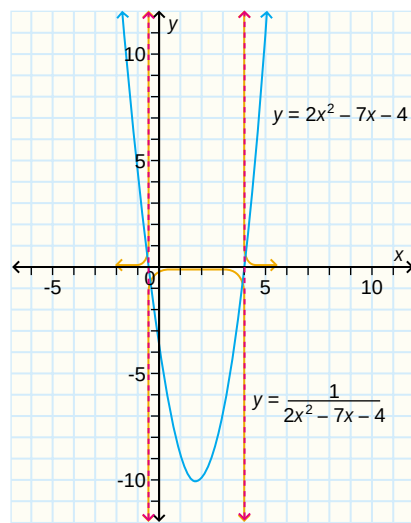
Given this information, you know that the graph of

$$y = \frac{1}{2x^2 - 7x - 4}$$

- has vertical asymptotes at $x = -\frac{1}{2}$ and $x = 4$
- has three parts to it
- crosses the parabola where $y = 1$ and $y = -1$
- has a high point between the asymptotes at $(1.75, -10.125)$ or about $(1.75, -0.1)$

Finally, the graph of $y = \frac{1}{2x^2 - 7x - 4}$

- has y-intercept $\frac{1}{-4} = -0.25$
- is close to the x-axis when the parabola is far from the x-axis and is far from the x-axis when the parabola is close to the x-axis



Practise, Apply, Solve 4.6

A

1. Which of the following tables display inverse variation? Explain your answer.

(a)

x	y
2	24
3	16
4	12
6	8

(b)

x	y
10	360
8	450
6	600
4	900

(c)

x	y
3	9
6	36
9	81
12	144

2. Write an equation for each relation in question 1.

3. i. Determine the reciprocal $y = \frac{1}{f(x)}$ for each of the following functions.

ii. Determine the equations of the vertical asymptotes of $y = \frac{1}{f(x)}$.

iii. Verify your results from (ii) using graphing technology.

(a) $f(x) = 2x$

(b) $f(x) = x + 5$

(c) $f(x) = x - 4$

(d) $f(x) = 2x + 5$

(e) $f(x) = -3x + 6$

(f) $f(x) = (x - 3)^2$

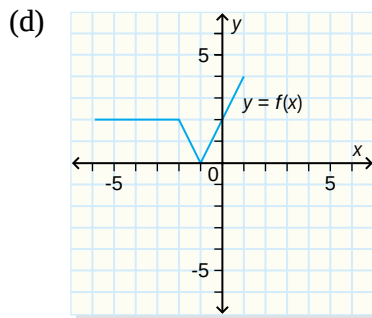
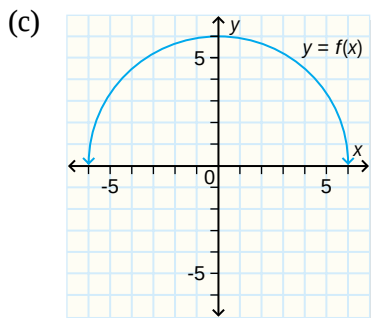
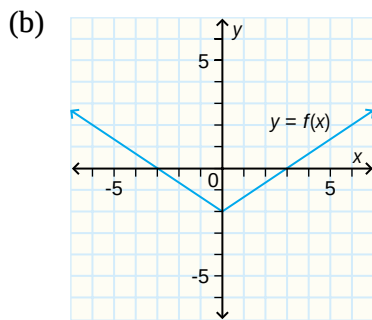
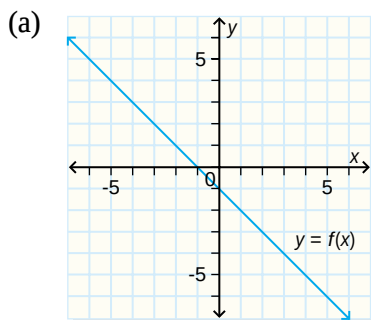
(g) $f(x) = x^2 - 3x - 10$

(h) $f(x) = 3x^2 - 4x - 4$

4. Copy and complete the table. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$.
Find equations for $y = f(x)$ and $y = \frac{1}{f(x)}$.

x	-4	-3	-2	-1	0	1	2	3	4	5	6	7
f(x)	16	14	12	10	8	6	4	2	0	-2	-4	-6
$\frac{1}{f(x)}$												

5. Sketch the graph of the reciprocal function for each of the following functions.



6. In the table, one value of y is incorrect. Sketch the graph to identify and correct the error. Determine an equation for the relationship between x and y .

x	3	4	5	6	7
y	105	87.5	63	52.5	45

B

7. The relationship between the time it takes for a car to accelerate to a given speed from rest and the acceleration is an inverse variation. It takes 6 s to reach a certain speed when the acceleration is 10 m/s^2 . Find the time it takes to reach the same speed when the acceleration is 12 m/s^2 . Write an equation to describe the relationship.

8. The current I , in amps (A), flowing through a light bulb can be given by $I = \frac{W}{V}$, where W is the power, in watts (W), and V is the voltage, in volts (V).

(a) How much current flows through a bulb marked 60 W, 100 V when it is lit?

(b) What total power could be obtained from a 110-V supply that is protected by a 15-A fuse?

(c) Describe the relationship between I and W for a fixed voltage supply.

9. (a) Sketch the graph of the function $f(x) = \frac{1}{x+2}$ by first sketching the graph of $f(x) = x+2$.

(b) State the domain and the range of $f(x) = \frac{1}{x+2}$. What is the inverse function?

- 10. Knowledge and Understanding:** Sketch each pair of graphs on the same axes. For each reciprocal function, state the domain and the range and find the equation of the inverse function.

(a) $y = x - 1$ and $y = \frac{1}{x - 1}$ (b) $y = 5 - x$ and $y = \frac{1}{5 - x}$

- 11.** Sketch each pair of graphs on the same axes. State the domain and estimate the range for each reciprocal function.

(a) $y = 2x - 5$, $y = \frac{1}{2x - 5}$

(b) $y = 3x + 4$, $y = \frac{1}{3x + 4}$

(c) $y = -x$, $y = \frac{1}{-x}$

- 12.** Sketch the graph of $y = \frac{1}{2x^2 + 5x - 3}$ by first graphing $y = 2x^2 + 5x - 3$.

- 13.** For each case, draw the graph of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same axes.

(a) $f(x) = x^2 - 4$

(b) $f(x) = (x - 2)^2 - 3$

(c) $f(x) = x^2 - 3x + 2$

(d) $f(x) = (x + 3)^2$

(e) $f(x) = x^2 + 2$

(f) $f(x) = -(x + 4)^2 + 1$

- 14. Communication:** Explain how you would use a sketch of the graph of $y = x^2 - 3x - 10$ to sketch the graph of $y = \frac{1}{x^2 - 3x - 10}$.

- 15.** For each function, draw $f(x)$, $\frac{1}{f(x)}$, and $f^{-1}(x)$ on the same axes. State the domain and the range.

(a) $f(x) = 2x + 1$

(b) $f(x) = x^2 - 1$

(c) $f(x) = \sqrt{x - 4} - 3$

- 16. Check Your Understanding:** Describe the properties of the function $f(x) = \frac{1}{x}$ and how it is related to the function $f(x) = x$. For what real-world situations would a function of the form $f(x) = \frac{k}{x}$ be a useful model? If you were given some data, how would you determine if the model was appropriate?

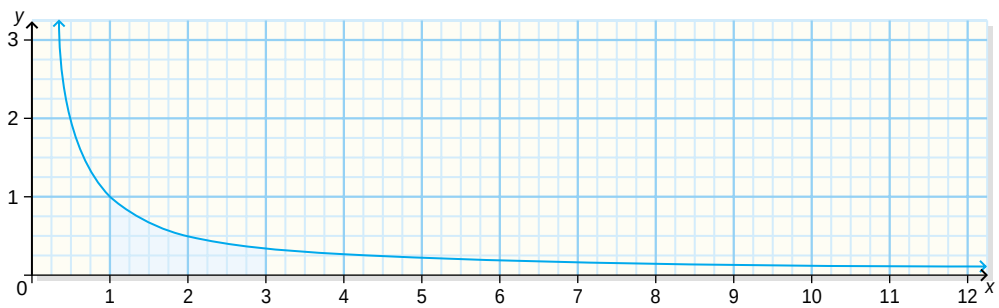


- 17.** The data in the table are modelled by a relation in the form $y = \frac{k}{x} + b$.

x	1	2	4	7	10
y	5.5	4	3.3	2.9	2.8

- (a) Calculate $\frac{1}{x}$ for each value of x in the table.
 (b) Plot the ordered pairs $\left(\frac{1}{x}, y\right)$ and fit a line through the points.
 (c) Find the slope and y -intercept. Write the equation of the original relationship.

- 18. Thinking, Inquiry, Problem Solving:** In the graph of $f(x) = \frac{1}{x}$, you can estimate the shaded area under the graph from $x = 1$ to $x = 3$ by counting the shaded squares.



Call this area $A(3)$. From the graph, $A(3) \doteq 1.1$ (17 small squares). Copy and complete the table.

x	1	2	3	4	5	6	7	8	9	10	12
$A(x)$											

You should find that $A(2) + A(3) \doteq A(6)$. Complete the following statements.

- (a) $A(3) + A(4) \doteq A(\blacksquare)$ (b) $A(2) + A(4) \doteq A(\blacksquare)$
 (c) $A(10) - A(2) \doteq A(\blacksquare)$ (d) $A(9) \doteq (\blacksquare) \times A(3)$

Use the table to estimate each of the following.

- i. $A(18)$ ii. $A(25)$ iii. $A(2.5)$ iv. $A(4.5)$

- 19. Application:** Determine the equation of the function in the graph.

