

You have seen that a function such as $f(x) = 2x + 5$ has a reciprocal function, $g(x)$, where $g(x) = \frac{1}{f(x)}$. In this case, $g(x) = \frac{1}{2x + 5}$. The expression $\frac{1}{2x + 5}$ is also an example of a rational expression. Rational expressions occur not only in rational functions but also in problem solving and further theoretical work in mathematics.

Think, Do, Discuss

1. Compare these two lists.

Expressions that are rational

$$\frac{x^2}{x^2 - 4}$$

$$x + 2 - \frac{4}{x}$$

$$4x^4 + 3x^2 - 2x - 1$$

$$\frac{1}{h} \left(\frac{1}{x + h} - \frac{1}{x} \right)$$

$$\frac{x^2 - 2x + 1}{x^2 - x} \div \frac{x - 1}{x}$$

Expressions that are not rational

$$\frac{x}{\sqrt{x - 1}}$$

$$1 - x - \frac{2}{\sqrt{x}}$$

$$3x^4 - 2x^{\frac{3}{2}} + 4x - 1$$

$$\frac{1}{h} \left(\frac{1}{\sqrt{x + h}} - \frac{1}{\sqrt{x}} \right)$$

$$\frac{x^{\frac{1}{2}} - 2}{x - 4} \div \frac{x^{\frac{1}{2}} + 2}{x^{\frac{1}{2}} + 2}$$

Given these examples, describe a rational expression. Explain your answer.

2. Rational expressions represent variable quantities and since they have variables in the denominator, you must ensure that the variables do not take values that would make the denominator equal to zero. Therefore, you must restrict the values of the variables. For example, $\frac{x}{x + 2}$ is defined only when $x \neq -2$. What are the restrictions for $\frac{5y}{x(y - 3)^2}$?
3. Simplifying rational expressions is the same as simplifying rational numbers. For example, $\frac{12}{16} = \frac{\cancel{4} \times 3}{\cancel{4} \times 4} = \frac{3}{4}$. How would you simplify the expression $\frac{6x - 9}{4x - 6}$?
4. When you simplify a rational number, the simplified number has exactly the same value as the original number. The two numbers are exactly equivalent and interchangeable. When you reduce a rational expression, the reduced expression must have the same restrictions as the original expression. For what values of x is the expression $\frac{(3x - 5)}{(x - 4)(3x - 5)}$ not defined? Explain your answer. You must state all the restrictions on a variable before reducing a rational expression. What is the reduced expression?

5. What are the restrictions for $\frac{12xy^3}{9x^2y}$? Simplify the expression.
6. What are the restrictions for $\frac{18x(x-2y)}{12x(2x-y)}$? Simplify the expression.
7. What is the first step in deciding whether the expression $\frac{3x^2 - 7x + 4}{x^2 - 1}$ is simplified? Simplify the expression, if possible, and state any restrictions.

Focus 4.8

Key Ideas

- A rational expression is a sum, difference, product, or quotient of terms, each of which could be written as a quotient of polynomials. A rational expression must have at least one rational term. No variable in a rational expression can be under a fractional exponent.
- Simplifying a rational expression is the same as simplifying a rational number. You may divide the numerator and denominator of a rational expression by any nonzero factor of the numerator or denominator.
- A rational expression cannot be defined when its denominator is 0.
- When stating or simplifying a rational expression, exclude any value(s) of the variables that would make the denominator 0. In other words, state the restrictions on the variables.
- An expression in the form $\frac{f(x)h(x)}{g(x)h(x)}$ is equal to $\frac{f(x)}{g(x)}$, provided $g(x) \neq 0$ and $h(x) \neq 0$.
- When a rational expression is simplified, it is often necessary to factor the numerator, or denominator, or both.

Example 1

For each of the following, state the restrictions on the value(s) of the variable(s).

(a) $\frac{6x^2z}{2xy}$

(b) $\frac{5x-15}{x^2-9}$

(c) $\frac{m^2+5m-6}{6m^3-5m^2-4m}$

Solution

For each case, find the value(s) of the variable(s) that make the denominator 0.

(a) For $2xy = 0$, $x = 0$ or $y = 0$. The restrictions are $x \neq 0$ and $y \neq 0$.

(b) Let $x^2 - 9 = 0$, then factor.

$$(x-3)(x+3) = 0$$

$$x = \pm 3$$

The restrictions are $x \neq \pm 3$.

(c) Let $6m^3 - 5m^2 - 4m = 0$, then factor.

$$m(6m^2 - 5m - 4) = 0$$

$$m(2m+1)(3m-4) = 0$$

$$m = 0 \text{ or } m = -\frac{1}{2} \text{ or } m = \frac{4}{3}$$

The restrictions are $m \neq -\frac{1}{2}, 0, \text{ or } \frac{4}{3}$.

Example 2

Simplify and state any restrictions.

(a) $\frac{3x - 12}{x^2 - x - 12}$

(b) $\frac{xy - x^2}{x^2y - y^3}$

Solution

(a) $\frac{3x - 12}{x^2 - x - 12} = \frac{3(x - 4)}{(x + 3)(x - 4)}, x \neq -3, 4$
 $= \frac{3}{x + 3}, x \neq -3, 4$

(b) $\frac{xy - x^2}{x^2y - y^3} = \frac{x(y - x)}{y(x^2 - y^2)}, y \neq 0$
 $= \frac{-x(x - y)}{y(x + y)(x - y)}, x \neq \pm y$
 $= \frac{-x}{y(x + y)}, y \neq 0, \text{ and } x \neq \pm y$

Example 3

An expression for calculating the drug dose for young children is $\frac{(t^3 - 2t^2 - 3t)a}{24t^2 - 72t}$,

where t is the age of the child in years and a is the adult dose in milligrams.

The adult dose for a pain-killing drug is 300 mg. What is the drug dose for

(a) a two-year-old child?

(b) a three-year-old child?

Solution

Calculating will be much easier after simplifying the formula.

Begin by factoring the numerator and the denominator.

$$\begin{aligned}\frac{(t^3 - 2t^2 - 3t)a}{24t^2 - 72t} &= \frac{t(t^2 - 2t - 3)a}{24t(t - 3)} \\ &= \frac{t(t - 3)(t + 1)a}{24t(t - 3)} \\ &= \frac{(t + 1)a}{24}, t \neq 0, 3\end{aligned}$$

(a) If $a = 300$ and $t = 2$, then $\frac{(2 + 1)300}{24} = 37.5$

The dose is 37.5 mg.

(b) Since $t \neq 3$, you cannot substitute the value 3 in the formula.

However, use $t = 2.9999$ or $t = 3.0001$ to estimate the dose.

In either case, $\frac{(2.9999 + 1)300}{24} \doteq 50$, or $\frac{(3.0001 + 1)300}{24} \doteq 50$.

The estimated dose for a three-year-old child is 50 mg.

Practise, Apply, Solve 4.8

A

1. For each rational expression, state the restrictions on the value(s) of the variable(s).

(a) $\frac{3x - 2}{x - 1}$

(b) $\frac{4y}{2y - 1}$

(c) $\frac{-3m^2n^2}{6m^2n}$

$$\begin{array}{lll} \text{(d)} \frac{-4x^2(x-3)}{2x(x-3)} & \text{(e)} \frac{2p-1}{(p-4)(p+2)} & \text{(f)} \frac{-8(a+b)}{2(a+b)} \\ \text{(g)} \frac{4(m+3)}{8m(m+3)} & \text{(h)} \frac{3(2x+1)}{5(2x-3)} & \text{(i)} \frac{3a(a+2)}{4a(3a+4)} \end{array}$$

2. (a) Factor $4x + 12$. (b) Factor $5x + 15$.
 (c) Use your answers from (a) and (b) to simplify $\frac{4x+12}{5x+15}$. What restrictions apply to x ?

3. Identify which of the following are equivalent to $\frac{a+b}{b-a}$.

$$\text{(a)} -\frac{a+b}{a-b} \quad \text{(b)} -\frac{a+b}{b-a} \quad \text{(c)} -\frac{a+b}{-(b-a)} \quad \text{(d)} -\frac{b+a}{a-b}$$

4. Simplify.

$$\begin{array}{lll} \text{(a)} \frac{x+y}{y+x} & \text{(b)} -\frac{(y-x)}{y-x} & \text{(c)} \frac{x-y}{-(y-x)} \\ \text{(d)} \frac{-(y+x)}{-(x+y)} & \text{(e)} \frac{-(y-x)}{x-y} & \text{(f)} \frac{x-y}{y-x} \end{array}$$

5. Simplify and state the restrictions.

$$\text{(a)} \frac{x-2}{(x+5)(x-2)} \quad \text{(b)} \frac{4y(3y-4)}{(3y-4)(3y+4)} \quad \text{(c)} \frac{(x-1)(3x+5)}{(x-1)(2x+1)}$$

6. (a) Factor $k^2 - 4$. (b) Factor $3k^2 + 5k - 2$.

(c) Simplify $\frac{k^2-4}{3k^2+5k-2}$. State the restrictions.

7. (a) Factor $6p^2 + 30p + 36$ as fully as possible.

(b) Factor $3p^2 - 3p - 18$ fully.

(c) Simplify $\frac{6p^2+30p+36}{3p^2-3p-18}$. State the restrictions.

8. Simplify each rational expression and state the restrictions on the variables.

$$\begin{array}{llll} \text{(a)} \frac{2b+8}{5b+20} & \text{(b)} \frac{3x^2-2x}{3x^2-6x} & \text{(c)} \frac{m^2-4m}{3m^2-12m} & \text{(d)} \frac{15x-5x^2}{x^2-3x} \\ \text{(e)} \frac{x^2-4}{x^2-5x+6} & \text{(f)} \frac{p^2-p-20}{3p-15} & \text{(g)} \frac{y^2-7y+12}{y^2-y-12} & \text{(h)} \frac{a^2+a-20}{a^2-11a+28} \end{array}$$

9. **Knowledge and Understanding:** Simplify and state the restrictions.

$$\text{(a)} \frac{36x^2y}{-16xy^2} \quad \text{(b)} \frac{6x+24}{4x+16} \quad \text{(c)} \frac{9m^2-4}{9m^2-3m-2} \quad \text{(d)} \frac{x^2-10xy+25y^2}{x^2-25y^2}$$

10. Simplify and state the restrictions.

$$\text{(a)} \frac{6m^2-2m-4}{4m^2-4} \quad \text{(b)} \frac{2a^2+5a-3}{2a^2-7a+3} \quad \text{(c)} \frac{40+6x-18x^2}{9x^2-9x-10} \quad \text{(d)} \frac{9x^2-1}{9x^3+6x^2+x}$$

11. **Communication:** What does the word *restrictions* mean in connection with a rational expression, and why must you state the restrictions? Explain using examples.

12. (a) What is the relation between the expressions $x - y$ and $y - x$?

(b) Simplify $\frac{x^2+xy-2y^2}{y^2-x^2}$ and state the restrictions on the variables.

13. Simplify and state the restrictions.

$$\text{(a)} \frac{4a^2-b^2}{b-2a} \quad \text{(b)} \frac{x^2-2x+1}{1-x} \quad \text{(c)} \frac{k^2-10k+25}{25-k^2}$$

14. Two expressions for calculating the drug dose for young children are $\frac{(2t^3 + t^2 - t)a}{48t^2 - 24t}$ and $\frac{(3t^2 - 2t)a}{3t^2 + 34t - 24}$, where t is the age of the child in years and a is the adult dose. If the adult dose for a pain-killing drug is 250 mg, then what is the dose, to the nearest unit, for

- (a) a one-year-old child?
(b) an eight-month-old child?



15. **Thinking, Inquiry, Problem Solving:** Find a simplified expression to represent the ratio $\frac{\text{surface area}}{\text{volume}}$ for a cylindrical can with height h and base radius r . What affects the ratio more, a change in the height or a change in the base radius? Justify your answer.

16. **Check Your Understanding:** Craig simplified $\frac{a^2 - 5b^2}{2a^2 - 9ab - 5b^2}$ to $\frac{a - 5b}{2a - b}$. Is Craig correct? If not, explain where he made his error.



17. Under what conditions is $\frac{-4x^3 - 7x^2 - 3x}{12x + 13x^2 - 4x^3} = \frac{2x^3 + x^2 - x}{2x^3 - 9x^2 + 4x}$ true?

18. **Application:** The concentration of pain-killing drug A in the bloodstream, t hours after it is taken orally, is given by $\frac{5t^4 - 10t^2}{t^5 - 4t}$, and the concentration of pain-killing drug B in the bloodstream, t hours after it is taken orally, is given by the expression $\frac{3t^2 - 12t}{2t^3 - 8t^2 + 5t - 20}$. Simplify both expressions. Which drug, A or B, needs to be taken more often? Justify your answer.



The Chapter Problem—Fundraising

- CP14.** If $C(x)$ is the cost of producing x items, then the average cost function is $AC(x) = \frac{C(x)}{x}$. Find the average cost function for your chosen supplier. Graph this function and state the domain.
- CP15.** How many T-shirts must be sold so that the average cost per T-shirt is less than \$5?
- CP16.** If $P(x)$ is the profit earned from producing and selling x items, then the average profit is modelled by the function $AP(x) = \frac{P(x)}{x}$. Find the average profit function for your chosen supplier. Graph this function and state the domain.
- CP17.** At what point is the average profit a maximum?