

Multiplying and dividing rational expressions is the same as multiplying and dividing rational numbers.

Think, Do, Discuss

- How do you multiply two rational numbers? Explain using the product $\frac{5}{48} \times \frac{72}{85}$ as an example. Give the answer as a rational number in lowest terms.
- At what point is it easier to remove the common factors, before multiplying or after? Explain.
- Apply the same steps to find the product $\frac{x(x+1)}{x^2-1} \times \frac{2(x-1)^2}{5x^2}$.
Look for common factors and reduce, where possible. Since you are working with rational expressions, what must you be careful to state at each step?
- How do you divide two rational numbers? Explain using the quotient $\frac{25}{28} \div \frac{10}{21}$ as an example. Give the answer as a rational number in lowest terms.
- Apply the same steps to find the quotient $\frac{4x^2}{x^2+6x-7} \div \frac{x}{x^2-1}$.
As before, look for common factors and reduce, where possible. Why are there more restrictions to include when dividing rational expressions?

Focus 4.9

Key Ideas

- To multiply rational expressions
 - ◆ factor the numerators and denominators
 - ◆ state all the restrictions on the variables
 - ◆ divide out any factors that are common to the numerator and denominator
 - ◆ multiply the numerators, then multiply the denominators, and write the result as a single rational expression
- To divide two rational expressions
 - ◆ factor the numerators and denominators
 - ◆ note any restrictions on the variables
 - ◆ take the reciprocal of the second rational expression and change $\div$ to $\times$
 - ◆ state any new restrictions

- ♦ remove any factors common to the numerator and denominator
- ♦ multiply the numerators and then multiply the denominators
- ♦ write the product as a single rational expression and state all restrictions
- If $f(x)$, $g(x)$, $m(x)$, and $n(x)$ are rational expressions, then

$$\frac{f(x)}{g(x)} \times \frac{m(x)}{n(x)} = \frac{f(x)m(x)}{g(x)n(x)}, \text{ provided } g(x) \neq 0 \text{ and } n(x) \neq 0$$

$$\frac{f(x)}{g(x)} \div \frac{m(x)}{n(x)} = \frac{f(x)}{g(x)} \times \frac{n(x)}{m(x)} = \frac{f(x)n(x)}{g(x)m(x)}, \text{ provided } g(x) \neq 0, n(x) \neq 0, \text{ and } m(x) \neq 0$$

Example 1

Simplify.

$$(a) \quad \frac{9x^2}{4xy} \times \frac{12xy^2}{3x}$$

$$(b) \quad \frac{2a+6}{7} \times \frac{3}{a^2-9}$$

$$(c) \quad \frac{2x^2+5x+2}{4x^2-8x-5} \times \frac{2x^2-11x+15}{3x^2+7x+2}$$

Solution

Follow these steps for multiplying rational expressions: factor, state restrictions, remove common factors, and then multiply.

$$(a) \quad \frac{9x^2}{4xy} \times \frac{12xy^2}{3x} = \frac{(3x)(3x)}{4xy} \times \frac{(4xy)(3y)}{3x} \quad \text{Remove the common factors.}$$

$$= \frac{3x}{1} \times \frac{3y}{1} \quad \text{Multiply.}$$

$$= 9xy, x \neq 0 \text{ and } y \neq 0$$

$$(b) \quad \frac{2a+6}{7} \times \frac{3}{a^2-9} = \frac{2(a+3)}{7} \times \frac{3}{(a-3)(a+3)}$$

$$= \frac{2}{7} \times \frac{3}{a-3}$$

$$= \frac{6}{7(a-3)}, a \neq \pm 3$$

$$(c) \quad \frac{2x^2+5x+2}{4x^2-8x-5} \times \frac{2x^2-11x+15}{3x^2+7x+2} = \frac{(2x+1)(x+2)}{(2x+1)(2x-5)} \times \frac{(2x-5)(x-3)}{(x+2)(3x+1)}$$

$$= \frac{1}{1} \times \frac{(x-3)}{(3x+1)}$$

$$= \frac{x-3}{3x+1}, x \neq -2, -\frac{1}{3}, -\frac{1}{2}, \frac{5}{2}$$

Example 2

Simplify.

$$(a) \frac{2x+8}{3x} \div \frac{4x+16}{2x^2}$$

$$(b) \frac{x^2+3x+2}{x^4-4x^2} \div \frac{x^2-x-2}{5x^3-9x^2-2x}$$

Solution

Follow these steps for dividing rational expressions: factor, state restrictions, take the reciprocal of the second expression and change $\div$ to $\times$, state additional restrictions, remove common factors, and then multiply.

(a) Factor.

$$\begin{aligned} \frac{2x+8}{3x} \div \frac{4x+16}{2x^2} &= \frac{2(x+4)}{3x} \div \frac{4(x+4)}{2x(x)} && \text{Multiply by the reciprocal.} \\ &= \frac{2\cancel{(x+4)}^1}{3\cancel{x}^1} \times \frac{2x\cancel{(x)}^1}{4\cancel{(x+4)}^1} && \text{Remove common factors.} \\ &= \frac{1}{3} \times \frac{x}{1} && \text{Multiply.} \\ &= \frac{x}{3}, x \neq 0, -4 \end{aligned}$$

$$\begin{aligned} (b) \frac{x^2+3x+2}{x^4-4x^2} \div \frac{x^2-x-2}{5x^3-9x^2-2x} &= \frac{(x+1)(x+2)}{x^2(x^2-4)} \div \frac{(x+1)(x-2)}{x(5x^2-9x-2)} \\ &= \frac{(x+1)(x+2)}{(x)(x)(x+2)(x-2)} \div \frac{(x+1)(x-2)}{x(5x+1)(x-2)} \\ &= \frac{1\cancel{(x+1)}^1\cancel{(x+2)}^1}{1\cancel{(x)}^1\cancel{(x)}^1\cancel{(x+2)}^1(x-2)} \times \frac{1\cancel{x}^1\cancel{(5x+1)}^1\cancel{(x-2)}^1}{1\cancel{(x+1)}^1\cancel{(x-2)}^1_1} \\ &= \frac{1}{x(x-2)} \times \frac{(5x+1)}{1} \\ &= \frac{5x+1}{x(x-2)}, x \neq -1, -\frac{1}{5}, 0, \pm 2 \end{aligned}$$

Example 3

Find the value of $\frac{3m^2-7m-6}{6m^2+3m} \times \frac{9m-6}{2m^2-5m-3} \div \frac{9m^2-4}{4m^2+4m+1}$ if $m = -0.2$.

Solution

Simplify the expression before substituting the value for m .

$$\begin{aligned} \frac{3m^2-7m-6}{6m^2+3m} \times \frac{9m-6}{2m^2-5m-3} \div \frac{9m^2-4}{4m^2+4m+1} \\ &= \frac{(3m+2)(m-3)}{3m(2m+1)} \times \frac{3(3m-2)}{(2m+1)(m-3)} \div \frac{(3m-2)(3m+2)}{(2m+1)(2m+1)} \\ &= \frac{1\cancel{(3m+2)}^1\cancel{(m-3)}^1}{1\cancel{3m}^1\cancel{(2m+1)}^1_1} \times \frac{1\cancel{3}^1\cancel{(3m-2)}^1}{1\cancel{(2m+1)}^1\cancel{(m-3)}^1_1} \times \frac{1\cancel{(2m+1)}^1\cancel{(2m+1)}^1}{1\cancel{(3m-2)}^1\cancel{(3m+2)}^1_1} \\ &= \frac{1}{m} \times \frac{1}{1} \times \frac{1}{1} \\ &= \frac{1}{m}, m \neq -\frac{2}{3}, -\frac{1}{2}, 0, \frac{2}{3}, 3 \end{aligned}$$

If $m = -0.2$, then $\frac{1}{m} = \frac{1}{-0.2} = -5$.

Practise, Apply, Solve 4.9

A

1. Simplify. Remember to state any restrictions.

(a) $\frac{3x}{8} \times \frac{2y}{9}$

(b) $\frac{6}{a} \times \frac{5a^2}{3b}$

(c) $\frac{9x^2y}{4y} \times \frac{12xy}{3x^2y}$

(d) $\frac{14pq}{5q^2} \times \frac{10q^3}{21pq^2}$

(e) $\frac{-8x^2}{10y} \times \frac{5y^3}{-2x^2}$

2. (a) $\frac{8a}{9} \div \frac{2b}{3}$

(b) $\frac{3m^2}{n} \div \frac{2m}{n}$

(c) $\frac{-56a^2b^2}{24a^3b} \div \frac{16ab^3}{18b}$

(d) $\frac{36pqr^2}{-26p^3qr} \div \frac{-24pq^2r}{39pr}$

(e) $\frac{-5a^3}{3b^2} \div \frac{-10a^2b}{3b^2}$

3. Simplify.

(a) $\frac{3x}{x-2} \times \frac{4(x-2)}{6x}$

(b) $\frac{5y(y+3)}{4y} \times \frac{(y-5)}{(y+3)}$

(c) $\frac{(2a-1)}{3a(a+4)} \times \frac{a(a+4)}{(2a-1)}$

(d) $\frac{10x}{x+2} \div \frac{5}{2(x+2)}$

(e) $\frac{4b(b+1)}{(b-1)} \div \frac{12(b+1)}{(b+4)}$

(f) $\frac{(3s+1)}{2s-1} \div \frac{3s(s+1)}{2s-1}$

4. Express each product in lowest terms.

(a) $\frac{12y}{3y-9} \times \frac{4y-12}{6y^2}$

(b) $\frac{6p-12}{5p+5} \times \frac{2p+2}{3p-6}$

(c) $\frac{x^2-y^2}{x^2-16} \times \frac{x-4}{x-y}$

5. Simplify each quotient.

(a) $\frac{k+1}{k^2-1} \div \frac{k}{k-1}$

(b) $\frac{x^2-4}{x+3} \div \frac{x-2}{x^2-9}$

(c) $\frac{1-3q}{2q+1} \div \frac{1-9q^2}{4q^2-1}$

6. **Communication:** Explain why there may be more restrictions when you divide rational expressions than when you multiply the same two expressions. Use an example to illustrate your explanation.

B

For questions 7 to 9, simplify each expression.

7. (a) $\frac{x^2+2x+1}{x} \times \frac{x^2-x}{x^2-1}$

(b) $(x+1) \times \frac{1}{3x^2-3}$

(c) $\frac{2x^2-x-3}{x^2-1} \times \frac{x^2+x-2}{2x^2+x-6}$

(d) $\frac{x^2+10x+25}{x^2-8x+16} \times \frac{x^2-16}{x^2-25}$

(e) $\frac{x^2+5x+6}{x^2-4} \times \frac{x^2-6x+8}{x^2-x-12}$

(f) $\frac{2x^2-5x-3}{x^2-9} \times \frac{2x^2+5x-3}{2x^2+5x+2}$

8. (a) $\frac{x^2-9}{x^2-49} \div \frac{x+3}{x+7}$

(b) $\frac{x^2-1}{2x^2+3x+1} \div \frac{x+1}{2x+1}$

(c) $\frac{3x+9}{2x^4} \div \frac{x^2+5x+6}{4x^2}$

(d) $\frac{x^2+4x-5}{x^2-3x} \div \frac{x^2-1}{x^2-6x+9}$

(e) $\frac{x^2-2x-15}{12x+9} \div \frac{x+3}{6}$

(f) $\frac{x-2}{x^3} \div \frac{x^2-3x+2}{x^2}$

9. (a) $\frac{x^2-1}{x^2+6x+9} \div \frac{x^2+2x-3}{x^2+6x+9}$

(b) $\frac{m^2+3m+2}{m^2-3m} \times \frac{3-m}{m^2+7m+6}$

(c) $(4x^2-9y^2) \div \frac{3xy+2x^2}{2}$

(d) $\frac{1+4z+4z^2}{z} \times \frac{3z^4}{12z^2-3}$

(e) $\frac{a^2+10a+25}{a^2+2a-15} \times \frac{a^2-5a+6}{a^2+3a-10}$

(f) $\frac{21p-3p^2}{16p^2+4p^3} \div \frac{14-9p+p^2}{12+7p+p^2}$

10. Knowledge and Understanding: Simplify.

$$(a) \frac{5x^2 + 15x}{x^2 - 2x - 15} \times \frac{x^2 - 7x + 10}{10x - 20} \quad (b) \frac{k^2 + k - 20}{k^2 - 16} \div \frac{k^2 + 3k - 10}{k^2 + 2k - 8}$$

11. If $f(x) = \frac{x^2 - 16}{4x - x^2}$, then find

- (a) $f(1)$ (b) $f(-1)$ (c) $f(5)$ (d) $f(-2)$
 (e) $f(3)$ (f) $f(-3)$ (g) which values of x are not allowed?

12. Application: An estimate of the cost, in billions of dollars, to keep the atmosphere in Canada x percent free of a chemical toxin is given by

$$C(x) = \frac{x - 2}{100x - x^2} \div \frac{x^2 + 18x - 40}{5x^2 + 100x}.$$

Find the cost of keeping the atmosphere

- (a) 90% free of the chemical toxin
 (b) 99% free of the chemical toxin
 (c) 99.9% free of the chemical toxin



13. Check Your Understanding: Charlene could not evaluate the expression $\frac{15x^2 + 7x - 2}{2x^2 - x - 15} \div \frac{6x^2 - 11x - 10}{4x^2 - 25}$ for $x = 2.5$. Simplify the expression and explain why.

C

14. Simplify.

$$(a) \frac{a^2 - b^2}{a^2 + b^2} \times \frac{a^2 + ab}{b^2 - a^2} \div a$$

$$(b) \frac{x^3 + 4x^2}{x^2 - 1} \times \frac{x^2 - 5x + 6}{x^2 - 3x} \div \frac{x^2 + 2x - 8}{x^2 - 1}$$

15. Find the value of each of the following if $x = 2$.

$$(a) \frac{x+1}{x-1} \times \frac{x+3}{1-x^2} \div \frac{(x+3)^2}{1-x}$$

$$(b) \frac{x^2 + 6x + 5}{x^2 + 7x + 12} \times \frac{x^2 + 2x - 8}{x^2 - 25} \div \frac{x^2 - x - 2}{x^2 - 2x - 15}$$

16. Thinking, Inquiry, Problem Solving

If $\frac{(x-1)f(x)}{3x^2 + 5x - 2} \times \frac{x^2 + 2x}{2x^2 - x - 1} \div \frac{x^2 g(x)}{6x^2 + x - 1} = \frac{3}{x}$, find a relationship between $f(x)$ and $g(x)$.

17. Simplify. State any restrictions on the variables.

$$(a) \frac{2x^2 - 3x - 2}{6x^2 - x - 2} \times \frac{3x^2 - 5x + 2}{4x^2 - 1} \div \frac{x^2 - 1}{6x^2 + 7x + 2}$$

$$(b) \frac{6a^2 - 7ab - 3b^2}{4a^2 + 19ab - 5b^2} \div \frac{9b^2 - 4a^2}{4a^2 - 1} \times \frac{8a^2 + 10ab - 3b^2}{4a^2 + 4a + 1}$$

$$(c) \frac{\frac{m^2 - mn}{6m^2 + 11mn + 3n^2} \div \frac{m^2 - n^2}{2m^2 - mn - 6n^2}}{\frac{4m^2 - 7mn - 2n^2}{3m^2 + 7mn + 2n^2}}$$