

It is not possible to divide by a complex number directly, because the denominator has two independent terms. Overcome this difficulty using the product property of conjugate complex numbers. In other words, recall that $z\bar{z}$ is always a real number.

Example 1

Express in the form $a + bi$.

(a) $\frac{2 + 3i}{5 - 2i}$

(b) $\frac{2}{1 - i}$

(c) $\frac{3 + i}{i}$

Solution

In each case, multiply by $\frac{\bar{z}}{z}$, where z is the complex number in the denominator. This is the same as multiplying by 1.

- (a) In this case, $5 + 2i$ is the complex conjugate of $5 - 2i$. Multiply the numerator and denominator by $5 + 2i$.

$$\begin{aligned} \frac{2 + 3i}{5 - 2i} &= \frac{2 + 3i}{5 - 2i} \times \frac{5 + 2i}{5 + 2i} \\ &= \frac{(2 + 3i)(5 + 2i)}{(5 - 2i)(5 + 2i)} \\ &= \frac{10 + 4i + 15i + 6i^2}{25 + 10i - 10i - 4i^2} \\ &= \frac{10 + 19i + 6(-1)}{25 - 4(-1)} \\ &= \frac{4 + 19i}{29} \\ &= \frac{4}{29} + \frac{19}{29}i \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{2}{1 - i} &= \frac{2}{1 - i} \times \frac{1 + i}{1 + i} \\ &= \frac{2(1 + i)}{(1 - i)(1 + i)} \\ &= \frac{2 + 2i}{1 - i^2} \\ &= \frac{2 + 2i}{1 - (-1)} \\ &= \frac{2 + 2i}{2} \\ &= 1 + i \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \frac{3 + i}{i} &= \frac{3 + i}{i} \times \frac{-i}{-i} \\ &= \frac{(3 + i)(-i)}{i(-i)} \\ &= \frac{-3i - i^2}{-i^2} \\ &= \frac{-3i - (-1)}{-(-1)} \\ &= \frac{1 - 3i}{1} \\ &= 1 - 3i \end{aligned}$$

Example 2

Find the reciprocal of each of the following.

(a) $1 - 2i$

(b) $a + bi$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{1}{1 - 2i} &= \frac{1}{1 - 2i} \times \frac{1 + 2i}{1 + 2i} \\ &= \frac{1 + 2i}{1 - 4i^2} \\ &= \frac{1 + 2i}{5} \quad \text{or} \quad \frac{1}{5} + \frac{2}{5}i \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{1}{a + bi} &= \frac{1}{a + bi} \times \frac{a - bi}{a - bi} \\ &= \frac{a - bi}{a^2 - b^2i^2} \\ &= \frac{a - bi}{a^2 + b^2} \quad \text{or} \quad \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \end{aligned}$$

Example 3

Solve for z .

(a) $(3 + 4i) = z(1 + i)$

(b) $(2 - i) + 3z = (1 + i)z$

Solution

(a) $(3 + 4i) = z(1 + i)$

$$\begin{aligned}\frac{(3 + 4i)}{(1 + i)} &= z \\ \frac{(3 + 4i)}{(1 + i)} \times \frac{(1 - i)}{(1 - i)} &= z \\ \frac{3 + i - 4i^2}{1 - i^2} &= z \\ \frac{3 + i + 4}{1 + 1} &= z \\ \frac{7 + i}{2} &= z \\ \frac{7}{2} + \frac{1}{2}i &= z\end{aligned}$$

(b) $(2 - i) + 3z = (1 + i)z$

$$\begin{aligned}2 - i + 3z &= z + zi \\ 2 - i &= -2z + zi \\ 2 - i &= z(-2 + i) \\ \frac{2 - i}{-2 + i} &= z \\ \frac{(2 - i)(-2 - i)}{(-2 + i)(-2 - i)} &= z \\ \frac{-4 + i^2}{4 - i^2} &= z \\ \frac{-4 + (-1)}{4 - (-1)} &= z \\ \frac{-5}{5} &= z \\ -1 + 0i &= z\end{aligned}$$

Focus 4.10U

Key Ideas

- Simplify a quotient of complex numbers by making the denominator a real number.
- To divide complex numbers, multiply the numerator and the denominator by the conjugate of the denominator.

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{(a + bi)(c - di)}{c^2 + d^2}\end{aligned}$$

Practise, Apply, Solve 4.10U

In these questions a and b are real numbers, unless stated otherwise.

A

1. Write the complex conjugate for each complex number.

(a) $1 - i$

(b) $-1 + 2i$

(c) $\sqrt{3} - i$

(d) $-2 + \sqrt{2}i$

(e) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(f) $x + yi$

2. Write the reciprocal for each complex number in question 1. Then express each reciprocal in the form $a + bi$.
3. Express in the form $a + bi$.

(a) $\frac{30}{3+i}$	(b) $\frac{2}{1+i}$	(c) $\frac{4i}{1-i}$	(d) $\frac{10i}{1-2i}$	(e) $\frac{2+3i}{1-i}$
(f) $\frac{3-i}{1+2i}$	(g) $\frac{4i-3}{3+2i}$	(h) $\frac{4+i}{i}$	(i) $\frac{-2+5i}{-i}$	(j) $\frac{x+yi}{x-yi}$
4. If $z_1 = \sqrt{3} + i$ and $z_2 = \sqrt{3} - i$, then express $\frac{z_1}{z_2}$ in the form $a + bi$.
5. If $z = 3 - i$ and $w = 1 + 2i$, then express $\frac{z}{2w}$ in the form $a + bi$.

B

6. If $z_1 = 1 - i$ and $z_2 = 7 + i$, then express $\frac{z_1 - z_2}{z_1 z_2}$ in the form $a + bi$.
7. Express $\frac{1-i}{(3-i)^2}$ in the form $a + bi$.
8. Compare dividing complex numbers with dividing real numbers. In what ways are the methods similar? different? Is it possible to apply the method for dividing complex numbers to dividing real numbers? Explain your answer using examples.
9. Is it necessary to state restrictions on the variable when dividing complex numbers? Give examples with your explanation.
10. (a) If $z = 3 + 4i$, then express $z + \frac{25}{z}$ in its simplest form.
 (b) If $z = a + bi$, then find the real and imaginary parts of $z + \frac{1}{z}$.
11. Solve for z and express in the form $a + bi$.

(a) $(5 - 2i) = z(1 - i)$	(b) $(3 + 2i) + z = (1 - i)z$
(c) $4z - z(3 + i) = -1 + 3i$	(d) $z(5 - 3i)^2 = 4 + 3i$
12. Find the real and imaginary parts of each of the following.

(a) $\frac{2}{3+i} + \frac{3}{2+i}$	(b) $\frac{1}{a+bi} - \frac{1}{a-bi}$
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13. For the complex numbers $z_1 = \frac{a}{1+i}$, $z_2 = \frac{b}{1+2i}$, where a and b are real numbers, $z_1 + z_2 = 1$. Determine the values of a and b .
14. The complex number z satisfies $\frac{z}{z+2} = 2 - i$. Find the real and imaginary parts of z .
15. **Check Your Understanding:** State the product property of complex conjugates. Show how to use this property to express a quotient of complex numbers as a single complex number in the form $a + bi$. Include an example.

C

16. Solve for the complex numbers z and w , in the form $a + bi$.

$$z + (1 + i)w = 1 - 2i$$

$$(1 + 2i)z - 2iw = 9 + 6i$$