

Extending Algebra Skills: Working with Polynomials

A polynomial is an algebraic expression in the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$, where $a_0, a_1, a_2, a_3, \dots, a_n$ are real numbers and the exponents are non-negative integers. A polynomial with exactly one term is called a **monomial**, a polynomial with two terms is a **binomial**, and a polynomial with three terms is a **trinomial**.

It is often necessary to combine two or more polynomials. For example, a profit function is obtained by subtracting a cost function from a revenue function. The revenue function, itself, is a product of the demand function and a number of items.

Some special products, which occur frequently when combining polynomials, are as follows:

- $(a + b)(a - b) = a^2 - b^2$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$

You will also frequently use these exponent laws.

- $a^m \times a^n = a^{m+n}$
- $(a^m)^n = a^{m \times n}$
- $(xy)^m = x^m y^m$

Adding and Subtracting Polynomials and Multiplying a Polynomial by a Monomial

To add or subtract polynomials, combine like terms. Apply the distributive property of multiplication over addition when multiplying a polynomial by a monomial.

Example 1

Simplify.

- (a) $(-3x^2 - x + 1) + (-x^2 + 2x - 1)$ (b) $(x^4 - 2x^3 - 3) - (2x^4 + x^2 + 2x - 1)$
 (c) $5 - 2[3x^2 + x(1 - x)]$ (d) $3x(x^2 + 4x - 2) - 3(x^2 - 5x)$

Solution

$$\begin{aligned} \text{(a)} \quad & (-3x^2 - x + 1) + (-x^2 + 2x - 1) \\ &= -3x^2 - x + 1 - x^2 + 2x - 1 \\ &= -4x^2 + x \end{aligned}$$

Remove brackets.
Combine like terms.

$$\begin{aligned} \text{(b)} \quad & (x^4 - 2x^3 - 3) - (2x^4 + x^2 + 2x - 1) \\ &= x^4 - 2x^3 - 3 - 2x^4 - x^2 - 2x + 1 \\ &= -x^4 - 2x^3 - x^2 - 2x - 2 \end{aligned}$$

Remove brackets.
Combine like terms.

$$\begin{aligned}
 \text{(c)} \quad & 5 - 2[3x^2 + x(1 - x)] \\
 &= 5 - 2(3x^2 + x - x^2) \\
 &= 5 - 2(2x^2 + x) \\
 &= 5 - 4x^2 - 2x \\
 &= 5 - 2x - 4x^2
 \end{aligned}$$

Expand $x(1 - x)$ using the distributive property.
Collect like terms inside the brackets.

Expand.

Arrange terms in ascending, or descending, powers of x .

$$\begin{aligned}
 \text{(d)} \quad & 3x(x^2 + 4x - 2) - 3(x^2 - 5x) \\
 &= 3x^3 + 12x^2 - 6x - 3x^2 + 15x \\
 &= 3x^3 + 9x^2 + 9x
 \end{aligned}$$

Expand.

Combine like terms.

Multiplying Polynomials

The process of multiplying is called **expanding**. When multiplying polynomials, multiply each term of one polynomial by each term of the other polynomial.

Example 2

Simplify.

$$\text{(a)} \quad (2 - 5y)(3 - 2y + y^2)$$

$$\text{(b)} \quad (3 - 4z - 5z^2)(2 - 3z + z^2)$$

$$\text{(c)} \quad 4(a - 5)(a + 4) - (a + 3)(a + 1)$$

$$\text{(d)} \quad 2(x - 3)(x + 3) - 4(x - 1)^2$$

Solution

(a) Apply the distributive property.

$$\begin{aligned}
 & (2 - 5y)(3 - 2y + y^2) \\
 &= 6 - 4y + 2y^2 - 15y + 10y^2 - 5y^3 \\
 &= 6 - 19y + 12y^2 - 5y^3
 \end{aligned}$$

Combine like terms.

$$\begin{aligned}
 \text{(b)} \quad & (3 - 4z - 5z^2)(2 - 3z + z^2) \\
 &= 6 - 9z + 3z^2 - 8z + 12z^2 - 4z^3 - 10z^2 + 15z^3 - 5z^4 \\
 &= 6 - 17z + 5z^2 + 11z^3 - 5z^4
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & 4(a - 5)(a + 4) - (a + 3)(a + 1) \\
 &= 4(a^2 - a - 20) - (a^2 + 4a + 3) \\
 &= 4a^2 - 4a - 80 - a^2 - 4a - 3 \\
 &= 3a^2 - 8a - 83
 \end{aligned}$$

Apply the distributive property.
Expand.
Combine like terms.

$$\begin{aligned}
 \text{(d)} \quad & 2(x - 3)(x + 3) - 4(x - 1)^2 \\
 &= 2(x^2 - 9) - 4(x^2 - 2x + 1) \\
 &= 2x^2 - 18 - 4x^2 + 8x - 4 \\
 &= -2x^2 + 8x - 22
 \end{aligned}$$

Apply the distributive property.
Expand.
Combine like terms.

Key Ideas

- In a polynomial expression, the numerical coefficients are real numbers and the exponents are non-negative integers.
- Some polynomials have special names.
 - ♦ A **monomial** is a polynomial with one term.
 - ♦ A **binomial** is a polynomial with two terms.
 - ♦ A **trinomial** is a polynomial with three terms.
- The degree of a polynomial in one variable is the value of the highest exponent of the variable in the polynomial.
 - ♦ A polynomial in one variable of degree 0 is called a **constant**, for example, 5 and -3.2 .
 - ♦ A polynomial in one variable of degree 1 is called **linear**, for example, $2x - 1$ and $4 - 7.2x$.
 - ♦ A polynomial in one variable of degree 2 is called **quadratic**, for example, $3x^2 + 2x + 1$ and $2 - 9x^2$.
 - ♦ A polynomial in one variable of degree 3 is called **cubic**, for example, $2x^3 - 5$ and $5 - 4x + 3x^2 - 2x^3$.
- To add or subtract polynomials, combine like terms.
- Apply the distributive property when multiplying polynomials.
 - ♦ To multiply a polynomial by a monomial, multiply each term in the polynomial by the monomial.
 - ♦ To multiply two polynomials, multiply each term of one polynomial by each term of the other polynomial.

Practise, Apply, Solve 4.12**A****1. Simplify.**

- $(x^2 - 2x - 1) + (3x^2 + x + 2)$
- $(x^2 - x + 5) + (-x^2 + x - 3)$
- $(-3x^2 - 2x + 2) + (2x^2 - x - 5)$
- $(x^3 - 4x^2 + 1) - (x^2 - 2x + 4)$
- $(2x^4 - 2x^2 - 7) - (3x^4 - x^3 + x^2 - 4)$
- $(x^5 + x^4 + x^3 - 4x^2 - x + 5) - (-x^4 - x^3 + 2x + 4)$
- $(-2x^2 + 5x - 3) + (x^2 - 6x + 1) - (-3x^2 - 2x - 4)$

2. Simplify.

- (a) $3x(x - 2) - 4x(x + 1)$
- (b) $-x(2x - 5) - 3x(4 - x)$
- (c) $2x^2(x + 3) + 4x^2(x - 1)$
- (d) $-2x(x^2 - 5x - 2) - x(2x^2 - x - 4)$
- (e) $-4x(x^3 - 3x^2 + 1) + x(3x^2 - 2x - 4)$
- (f) $2x^2(3x^3 - 5x + 2) - x^3(2x^3 + 2x - 5)$

3. State the degree of each simplified polynomial in question 2.

4. Simplify.

- (a) $4[2x - (3 - 5x)] + 3[3x - (4 + x)]$
- (b) $5[x - (2x + 3)] - 2[4x - (x + 1)]$
- (c) $x[5x - (3x - 8)] + 2x[3x - (4 - x)]$
- (d) $-3x[x - (5 - 2x)] - x[-2x - (x - 6)]$
- (e) $2a[a - 2a(a - 2)] + 3a[a + 4a(1 - a)]$
- (f) $-a[5a - a(1 - 2a)] - 3a[a - 4a(a + 1)]$

5. Simplify.

- (a) $(x + 2)(x^2 - x - 2)$
- (b) $(1 - x)(x^3 - x^2 + 3x)$
- (c) $(x^2 + 1)(x^2 - 3x + 5)$
- (d) $(2 - x^2)(3 + x^2 - x^3)$
- (e) $(x^2 - 3)(x^3 - 3x^2 + 2x - 1)$
- (f) $(1 - x^3)(2 + 3x - 6x^2 + 2x^3 - x^4)$

B

6. Simplify.

- (a) $(x + 3)(x - 1) - (x - 5)(x - 2)$
- (b) $2(x + 1)(x - 4) - 4(x - 2)(x + 2)$
- (c) $3(2 - x)(x - 4) + (x - 1)^2$
- (d) $2(3x - 2)^2 - 4(x - 1)(2x + 5)$
- (e) $(2x^2 - 1)^2 - (4 - x^2)^2$
- (f) $2x(5x^2 - 2)^2 - 3x(2 - x^2)^2$

7. Simplify.

- (a) $(x - 1)(x + 3)(x + 1) + (2x - 1)(x - 2)$
- (b) $(x - 3)(x + 2)(x + 3) - (3 - x)^2$
- (c) $(2a + 7)^2(a - 2) + 2(a^2 - 1)(4a + 3)$
- (d) $(t - 1)(2t - 1)(3t + 4) - (1 - t)^2(t + 4)$
- (e) $3(x - 2)(x + 3)^2 - (3x - 1)(2x + 1)^2$
- (f) $(x - 1)^2(x + 3)^2 - (x + 1)^2(x - 2)^2$

8. Simplify.

(a) $(x^2 - 3x + 5) - (2x^2 - 5x - 7)$

(b) $x(2x - 3)^2 - 2x(x - 4)$

(c) $(1 - x + x^2 - x^3 - x^4)(1 + x)$

9. Jake wrote the following.

$$\begin{aligned}(x + 1)^2 - (x + 2)(x - 3) &= x^2 + 2x + 1 - x^2 - x - 6 \\ &= x - 5\end{aligned}$$

What mistake(s) did he make? Write a correct simplification, and explain how to avoid making Jake's error.

10. Expand each expression.

(a) $(2a - 3)(a^3 - 2a^2 + a - 1)$

(b) $(y^2 + 3y - 5)(2y^2 - y + 6)$

(c) $(x^2 - x + 1)^2$

(d) $(x - 1)^2(x + 3)^2$

11. **Check Your Understanding:** Create an expression that contains the difference of a product of two binomials and a product of a monomial and a trinomial. Simplify your expression and state the degree of the resulting polynomial.



12. Simplify.

(a) $(a^2 + 2a - 1)^2 + (a^2 - 2a + 1)^2$

(b) $(a^2 - ab + b^2)^2 - (a^2 - 2ab + b^2)^2$

13. For any five consecutive integers, prove that

(a) the square of the middle integer is always 4 more than the product of the first and fifth integers

(b) the product of the second and fourth integers is always 3 more than the product of the first and fifth integers

14. If $x + y + z = 1$ and $x^2 + y^2 + z^2 = 3$, then find the value of $xy + xz + yz$.

15. If $5x^4 - 14x^3 + 18x^2 + 40x + 16 = (x^2 - 4x + 8)(ax^2 + bx + c)$, then find a , b , and c and all solutions of the equation $5x^4 + 18x^2 + 16 = 14x^3 - 40x$.

16. For $x^3 - 1 = (x - 1)(ax^2 + bx + c)$, find the values of a , b and c . Find the three roots of the equation $x^3 = 1$.