

Loci and Conics

Analytic Geometry: Lines and Line Segments

Finding the Slope of a Line Given Two Points

The slope, m , of the line between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} \\ = \frac{\Delta y}{\Delta x} \quad \text{or} \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

Finding the Length of a Line Segment

Use the formula $d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ or $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ where $A(x_1, y_1)$ and $B(x_2, y_2)$ are the endpoints of the line segment.

Finding the Midpoint of a Line Segment

The coordinates of the midpoint of the line segment joining points $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Finding the Equation of a Line Given the Slope and a Point

Use the slope– y -intercept form of the line.

$$y = mx + b$$

1. Substitute the slope.
2. Substitute the coordinates of the point.
3. Solve for b .
4. Express the equation in the form $y = mx + b$.

Finding the Equation of a Circle Centred at the Origin

The equation of a circle centred at $(0, 0)$ with radius r is $x^2 + y^2 = r^2$.

Determining if Lines Are Parallel or Perpendicular

Compare the slopes.

Two lines are parallel if they have the same slope; that is, $m_1 = m_2$.

Two lines are perpendicular if their slopes are negative reciprocals; that is, if $m_1 = -\frac{1}{m_2}$.

Practice

- Determine the slope and length of the line segment through each pair of points.
 - $(3, 1), (-5, 2)$
 - $(-6, 4), (-5, -3)$
 - $\left(\frac{3}{4}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{7}{8}\right)$
 - $(0.5, -3.3), (4.5, 4.7)$
- Determine the equation of the line
 - with slope $-\frac{2}{3}$ that passes through $(5, -2)$
 - with slope $\frac{3}{5}$ and y -intercept 6
 - that passes through $(4, -2)$ and $(-3, -7)$
 - with x -intercept 5 and y -intercept -8
- Determine the equation of the circle centred at the origin with each radius.
 - $r = 5$
 - $r = \frac{8}{3}$
 - $r = 0.3$
- A circle centred at the origin has a radius of 13 units. State all values of y when $x = 5$.
- Each pair of coordinates represents points on a line. Determine whether the lines are parallel, perpendicular, or neither.
 - $(1, -4), (5, 4)$ and $(3, 15), (-2, 5)$
 - $(-3, -19), (4, 23)$ and $(6, 1), (-18, 5)$
 - $(15, 16), (-3, 4)$ and $(2, -7), (-4, 2)$
 - $(0, 7), (-3, 1)$ and $(2, -2), (-4, -5)$
 - $(2, 1), (-2, 13)$ and $(5, -19), (-1, -1)$
- Identify the lines that are parallel within each grouping.
 - $l_1: 2x + 3y - 15 = 0$
 $l_2: 2x - 3y = 18$
 $l_3: 3y = -2x - 21$
 - $l_1: 4y = 3x + 8$
 $l_2: 3x = 4y - 16$
 $l_3: 3x + 4y = 12$
- Identify the lines that are perpendicular within each grouping.
 - $l_1: x + 2y - 10 = 0$
 $l_2: 6x - 3y - 21 = 0$
 $l_3: 2x - y - 7 = 0$
 - $l_1: 3y = 4x + 15$
 $l_2: 8y = 3x - 8$
 $l_3: 3y + 4x = 15$
- The end points of the diameter of a circle are $(10, 24)$ and $(-10, 24)$.
 - What are the coordinates of the centre of the circle?
 - What is the radius of the circle?
 - Determine the equation of the circle.
- The end points of the chord on a circle are $(-2, 1)$ and $(5, -3)$. Find the equation of the perpendicular bisector of the chord.
- The vertices of $\triangle ABC$ are $A(3, 4)$, $B(4, -3)$, and $C(-4, -1)$. Determine the equation of the altitude from vertex B to the line AC .

Solving a System of Linear Equations

Many kinds of situations can be modelled using linear equations. When two or more linear equations are used to model a problem, they are called a linear system of equations. Point $P(x_1, y_1)$ is the intersection point of the linear equations in the system. Point P is called the solution to the linear system and satisfies all equations in the system.

Solving a Linear System Graphically

Linear systems can be solved graphically, although this method does not always yield an exact solution.

Example 1

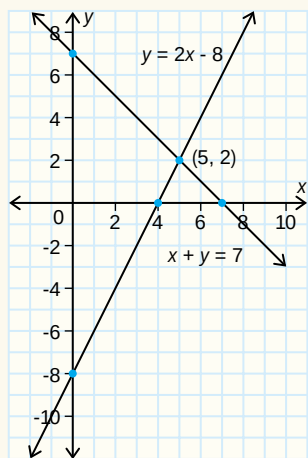
Solve the system graphically.

$$y = 2x - 8 \quad \textcircled{1}$$

$$x + y = 7 \quad \textcircled{2}$$

Solution

Draw both graphs on the same axes and locate the point of intersection.



Point $(5, 2)$ appears to be the point of intersection. Verify this result algebraically by substituting $(5, 2)$ into equations $\textcircled{1}$ and $\textcircled{2}$.

In equation $\textcircled{1}$,

L.S.	R.S.
y	$2x - 8$
$= 2$	$= 2(5) - 8$
	$= 2$

Therefore $\text{L.S.} = \text{R.S.}$

In equation $\textcircled{2}$,

L.S.	R.S.
$x + y$	7
$= 5 + 2$	
$= 7$	

Therefore $\text{L.S.} = \text{R.S.}$

Solving a System of Linear Equations Algebraically

Linear systems can also be solved using algebra. Algebra can be used in two ways to solve a linear system, elimination and substitution. The following example will be solved both ways. These methods always yield exact solutions.

Example 2

Solve the system of linear equations algebraically.

$$3x + 2y + 24 = 0 \quad \textcircled{1}$$

$$5y + 2x = -38 \quad \textcircled{2}$$

Solution by Elimination

$3x + 2y = -24 \quad \textcircled{3}$ $2x + 5y = -38 \quad \textcircled{4}$	Express both equations in the form $ax + by = c$.
To eliminate x, multiply equation $\textcircled{3}$ by 2 and equation $\textcircled{4}$ by 3. $6x + 4y = -48 \quad \textcircled{3} \times 2 \rightarrow \textcircled{5}$ $6x + 15y = -114 \quad \textcircled{4} \times 3 \rightarrow \textcircled{6}$	Choose a variable to eliminate from both equations. Multiply each equation by a number that gives the same or opposite coefficient for that variable in both equations
Equation $\textcircled{5}$ – Equation $\textcircled{6}$ $6x + 4y = -48 \quad \textcircled{5}$ $6x + 15y = -114 \quad \textcircled{6}$ $6x - 6x + 4y - 15y = -48 - (-114)$	Add or subtract like terms in the system to eliminate the chosen variable.
$-11y = 66$ $y = -6$	Simplify the resulting equation and solve for the remaining variable.
Substitute $y = -6$ into equation $\textcircled{1}$. $3x + 2(-6) + 24 = 0 \quad \textcircled{1}$ $3x = -24 + 12$ $3x = -12$ $x = -4$	Determine the other variable by substituting the solved variable into equation $\textcircled{1}$ or $\textcircled{2}$.

Solution by Substitution

Solve for y in equation ①. $3x + 2y + 24 = 0 \quad \text{①}$ $2y = -24 - 3x$ $y = -12 - \frac{3}{2}x$	Choose one of the equations and isolate one of its variables by expressing that variable in terms of the other variable.
Substitute $y = -12 - \frac{3}{2}x$ into equation ②. $5y + 2x = -38 \quad \text{②}$ $5\left(-12 - \frac{3}{2}x\right) + 2x = -38$ $-60 - \frac{15}{2}x + 2x = -38$ $\frac{-15x + 4x}{2} = -38 + 60$ $\frac{-11x}{2} = 22$ $x = 22\left(-\frac{2}{11}\right)$ $x = -4$	Substitute the expression that you determined in place of the corresponding variable in the other equation.
Substitute $x = -4$ into equation ①. $3x + 2y + 24 = 0 \quad \text{①}$ $3(-4) + 2y + 24 = 0$ $-12 + 2y + 24 = 0$ $2y = -12$ $y = -6$	Determine the other value by substituting the solved value into equations ① or ②.

Both methods yield the same solution of $(-4, -6)$.

Practice

- Determine which ordered pair satisfies both equations.
 - $x + y = 5$ $(4, 1), (2, 3),$
 $x = y + 1$ $(3, 2), (5, 4)$
 - $x + y = -5$ $(3, -6), (10, -5),$
 $y = -2x$ $(5, -10), (-3, -2)$
- Solve the system by drawing the graph.
 - $3x + 4y = 12$
 $2x + 3y = 9$
 - $x + y = -4$
 $2x - y = 4$
 - $x - 3y + 1 = 0$
 $2x + y - 4 = 0$
 - $x = 1 - 2y$
 $y = 2x + 3$
- Solve for the variable indicated.
 - $x - 3y = 12$, for x
 - $x - 4y = 15$, for x
 - $2x + y = 12$, for y
 - $8x - y = 4$, for y
 - $2y - 2x = 18$, for x
 - $6x - 3y = 4$, for y
 - $15x - 12y + 14 = 0$, for x
 - $5x + 2y + 10 = 0$, for y
 - $\frac{1}{2}x + \frac{2}{3}y = 8$, for y
- For each linear system, state which variable could be eliminated more easily.
 - $x + y = 4$
 $x - 3y = 5$
 - $3x - 2y = 4$
 $2x + 2y = 5$
 - $3x - 2y = 7$
 $5x + 7 = -2y$
 - $3y - 2x + 9 = 0$
 $14 + 7y = 2x$
- Determine the coordinates of the point of intersection using substitution.
 - $3p + 2q - 1 = 0$
 $p = q + 2$
 - $2m - n = 3$
 $m + 2n = 24$
 - $2x + 5y + 18 = 0$
 $x + 2y + 6 = 0$
 - $6g - 3h = 9$
 $4g = 5 + 3h$
- Determine the coordinates of the point of intersection using elimination.
 - $2m + 5n = 8$
 $5m - 2n = 20$
 - $3u + 21 = 5v$
 $4v + 6 = -9u$
 - $2x + 3y = 18$
 $5x - 4y = -1$
 - $3x + 2y - 22 = 0$
 $5x = 4y - 22$
- Solve each system of equations using substitution or elimination as appropriate.
 - $2g + h = 3$
 $3g + 2h = 5$
 - $m - 3n - 11 = 0$
 $2m = 6 - 10n$
 - $3x + y = 9$
 $x - 2y = -7$
 - $2p = q + 5$
 $q - 3p = 0$