

You can create interesting geometric designs with a Spirograph™. This device is a plastic frame with a large circle cut out of the middle.

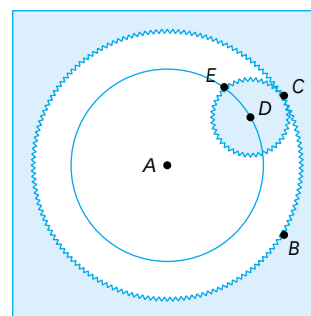
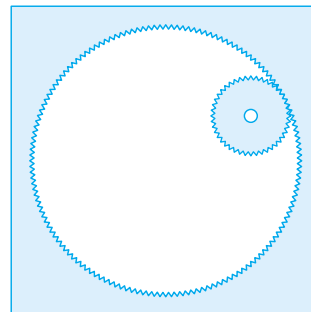
The inside of the larger circle has “teeth” so that other smaller circles, or other shapes, with “teeth” can be turned inside the larger circle. You can make a design by inserting a pen in one of the holes of the smaller circle and pushing it around the inside of the large circle.

If you have used a Spirograph™ before, you may have wondered how or why different designs were created. Start with a simple example. If you were to place your pen in the hole at the centre and push the smaller circle around the inside of the large circle, what shape would you create?

In this case, you would create another circle. The circle is a set of points such that each point is a fixed distance from another larger circle. This set of points is defined as the locus of points that are a fixed distance from the outer circle.

A locus is a set of points that satisfy a *given condition* or conditions.

Specific conditions that define a geometric model is called a locus definition. A locus definition tells us the requirements for drawing, graphing, constructing, or creating an algebraic expression for our mathematical object.



Part 1: Locus Definitions

Think, Do, Discuss

1. If you created a circle with a Spirograph™, then you would certainly recognize that this is not the usual way of drawing a circle. You usually draw a circle with compasses, by placing one tip of a compasses' arm at a fixed point and then rotating the other arm at a fixed radius about the fixed point. Construct a circle using compasses and then define the circle using a locus definition. You might begin the definition with, “A circle is the set of points that are”
2. Different definitions can describe the same geometric figure. What figure would be described by the locus of points that are equidistant from the centre and edge of a circle? What specific characteristics would this figure have?
3. The typical locus definition of a circle is the set of points that are equidistant from a given point. In the three-dimensional plane, what geometric figure is the locus of points that are the same distance from a given point?

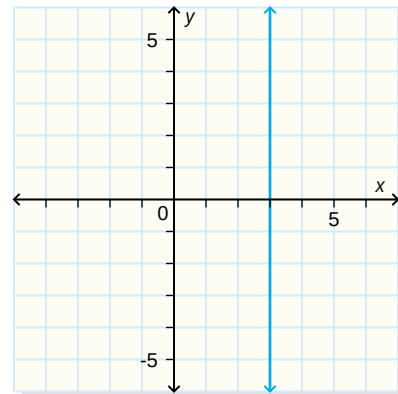
4. How would you construct a set of points, in the two-dimensional plane, that are equidistant from two given points? Use *The Geometer's Sketchpad* to explore this question. Where would these points lie? Describe the set of points geometrically.
5. Consider the locus of points that are equidistant from two lines. How would you describe this locus?
6. The locus definition may also describe a set of points that fall within a region or area. Imagine that you are with your class on a nature hike. You approach a swampy area, but you cannot tell what is swamp and what is solid ground. There are two posts in the ground. Your teacher tells you that, if the angle between you and the two posts is more than 90° , then you will walk on swampy ground. If the angle is less than 90° , then you will not walk on swampy ground. Draw a diagram or use *The Geometer's Sketchpad* to explore the problem and determine the edges of the swamp. Explore what would happen for an angle other than 90° .

Part 2: Locus Definitions That Lead to Algebraic Expressions

Recall that a locus is a set of points that satisfy a given condition or conditions. The set of points can represent a geometric figure, an area or region, or an equation.

Think, Do, Discuss

1. The adjacent diagram shows the locus of points that are 3 units to the right of the y -axis. What is the equation of this locus?
2. Describe the locus of points that are 5 units from the x -axis. Draw a diagram to help you.
3. What is the equation, or equations, for this locus of points?
4. Draw a sketch of the locus of points that are 12 units from the y -axis. Describe this locus geometrically and algebraically.
5. Describe all the points that are less than 4 units from the x -axis. Create a geometric model and an algebraic model.
6. Describe the locus of points that are 4 units from the origin. If possible, write the equation for the locus.



Key Ideas

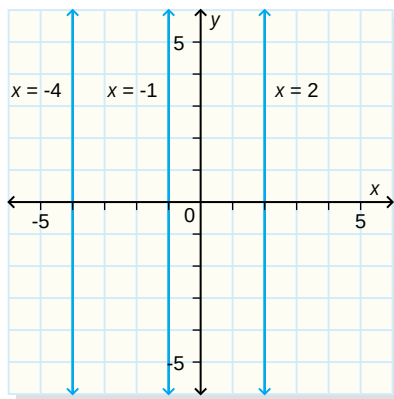
- A locus is a set of points that satisfy a *given condition* or conditions.
- You can describe many geometric figures with a locus definition.
- A locus definition may describe a geometric figure, a region, or a relation defined by an equation.
- The locus of points that are equidistant from a fixed point is a circle.
- The locus of points that are equidistant from two fixed points is the right bisector of the segment joining those points.

Example 1

Describe the locus of the points that are 3 units from the line $x = -1$. Give the equation of the locus.

Solution

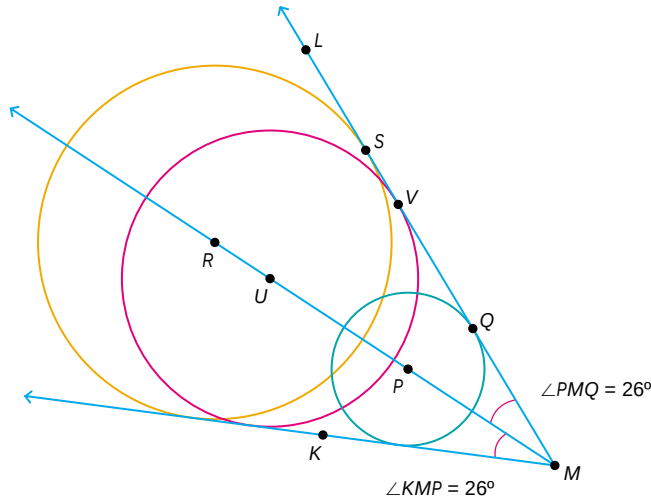
Draw a diagram. Points that are 3 units from the line $x = -1$ are on both sides of the line. The points to the left of the line form a straight line with the equation $x = -4$. The points to the right form a straight line with the equation $x = 2$.

**Example 2**

Construct an acute angle and then draw several inscribed circles in the angle. Determine the locus of the circles' centres.

Solution

You could draw the figures using *The Geometer's Sketchpad*. By constructing several inscribed circles in $\angle KMQ$, you will see that the centres of the circles lie on the angle bisector of the angle. The locus of the circles' centres is the angle bisector of the angle.

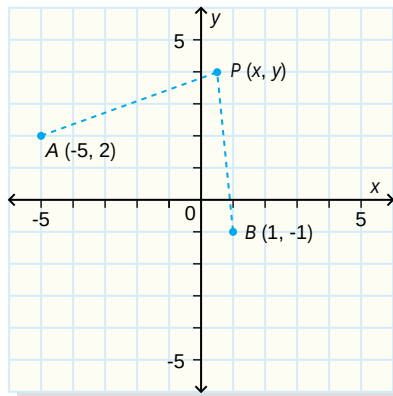


Example 3

Find the equation of the locus of points that are equidistant from points $A(-5, 2)$ and $B(1, -1)$.

Solution

Draw a diagram and let $P(x, y)$ be any point on the locus.



Point P is the same distance from A and from B . Then $PA = PB$.

Using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,

$PA = \sqrt{(x + 5)^2 + (y - 2)^2}$, and $PB = \sqrt{(x - 1)^2 + (y + 1)^2}$.

Since $PA = PB$, then

$$\sqrt{(x+5)^2 + (y-2)^2} = \sqrt{(x-1)^2 + (y+1)^2}$$

Square both sides.

$$(x+5)^2 + (y-2)^2 = (x-1)^2 + (y+1)^2$$

Expand.

$$x^2 + 10x + 25 + y^2 - 4y + 4 = x^2 - 2x + 1 + y^2 + 2y + 1$$

Simplify.

$$x^2 + 10x + 25 + y^2 - 4y + 4 - x^2 + 2x - 1 - y^2 - 2y - 1 = 0$$

$$12x - 6y + 27 = 0$$

The equation for the locus of points is $12x - 6y + 27 = 0$.

Practise, Apply, Solve 7.1U

A

- Draw a diagram that represents the locus in each of the following situations.
 - sitting in a seat on a Ferris wheel as it rotates
 - going up 15 floors in an elevator
 - going down an escalator
 - walking from the corner of a room so that you are always the same distance from a pair of adjacent walls
- What is the locus of each of the following? Draw a diagram to help you.
 - all points that are 10 cm from a fixed point
 - all points that are 3 cm above a line
 - all points that are 2 cm from a parabola
 - all points that are 1 cm above the surface of a sphere
- For each set of points, describe the locus geometrically. Then graph and create an algebraic expression.
 - 5 units from the line $y = 2$
 - 3 units from the line $x = -2$
 - 2 units from the line $y = 2x - 3$
 - 4 units from the line $y = -3x + 2$
- Knowledge and Understanding**
 - Find the equation for the locus of points that are 5 units from the origin.
 - In right triangle ABC , the hypotenuse, BC , is 10 units. Determine an equation for the locus of points for the vertex A . What will a trace of these points look like?
- Describe, geometrically, the locus of points that are
 - 6 units from $(3, -1)$
 - 3 units from $(0, 3)$
 - 4 units from $(2, 3)$

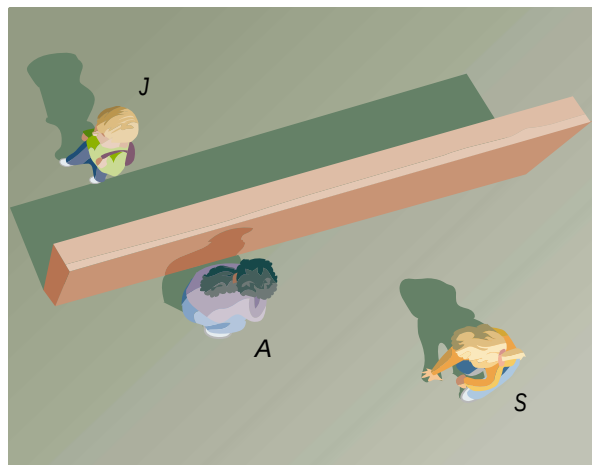


B

6. i. Sketch each locus on the x - y plane.
ii. Determine the equation(s) of each locus.
- (a) the points 4 units above the line $y = 3x + 6$
 - (b) the points 3 units below the line $2x + 4y = -8$
 - (c) the points 4 units from the circumference of the circle with centre $(0, 0)$ and radius 3 units
 - (d) the points that lie directly 5 units to the right of the curve $y = x^2$
7. Describe and create an equation for each locus.
- (a) the points that are equidistant from points $(0, 2)$ and $(0, 6)$
 - (b) the points that are equidistant from points $(0, 0)$ and $(2, 4)$
 - (c) the points that are equidistant from points $(3, 3)$ and $(-3, -3)$
 - (d) the points that are equidistant from points $(1, 5)$ and $(-2, 6)$
 - (e) the set of points that create a line parallel to the line $y = 2x + 6$, and the y -intercept is 3
 - (f) the set of points that create a line perpendicular to the line $y = 3x + 7$, and the locus passes through the origin
8. Sketch each locus and determine the equation. Let $P(x, y)$ be any point.
- (a) The sum of the coordinates of any point is 8.
 - (b) The difference between the coordinates of any point is 3.
 - (c) The average of the coordinates of any point is 2.
 - (d) The product of the coordinates of any point is 20.
9. **Communication:** Create a locus definition in words for each of the following.
- (a) $y = x$
 - (b) $x^2 + y^2 = 5$
 - (c) $x + y = 5$
 - (d) $x = -2$
 - (e) $y = 5$
 - (f) $y = \sqrt{16 - x^2}$
10. Four rods, all of equal length, are hinged at their end points to form the roaming rhombus *ROAM*. Keeping *RO* fixed, make *MA* move. What is the locus (or path) of point *M*? If *Y* is the midpoint of *MA*, what is the locus of *Y*?
11. The slope of the line joining any point, $P(x, y)$, with point $(4, 2)$ has a slope of $\frac{3}{4}$. Determine the equation of the locus.
12. Determine the equation, or equations, of the locus whose points are equidistant from the x - and y -axes.
13. Any point, $P(x, y)$, of a locus lies 2 units to the right and 5 units below the curve $y = x^2 + 2x + 4$. Determine the equation of the locus.
14. For any point, $P(x, y)$, of a locus, the sum of the squares of the coordinates is 9.
- (a) Graph the locus.
 - (b) Determine the equation of the locus.

15. (a) Create a set of conditions that define a locus.
(b) Use your definition to graph the locus.
(c) Determine the equation of the locus.

16. **Application:** Amy and Sara are playing in the schoolyard at recess and are hiding behind a high wall as shown. Joaquin is on the other side of the wall and is looking for them.

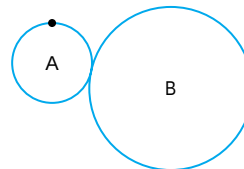


Show each of the following in a diagram.

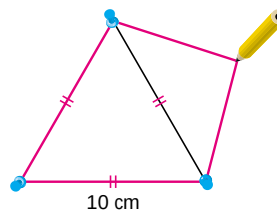
- (a) the points from which Joaquin can see neither Amy nor Sara
 - (b) the points from which Joaquin can see Sara, but not Amy
 - (c) the points from which Joaquin can see both Amy and Sara
17. **Check Your Understanding:** Describe and find the equation for the locus of points that are
 - (a) equidistant from the origin and the x -axis.
 - (b) equidistant from $(-2, 1)$ and $(6, 3)$
 - (c) equidistant from $y = 4$ and $y = -2$
 - (d) equidistant from points $A(3, 4)$ and $B(7, 8)$

C

18. Imagine a smaller circle, A, rolling around a larger, fixed circle, B. Describe the path of a specific point on A. You could use *The Geometer's Sketchpad* to explore this problem.



19. **Thinking, Inquiry, Problem Solving:** Create an equilateral triangle with three pushpins placed 10 cm apart in a bulletin board, covered with paper, or in a piece of corrugated cardboard. Place a 34-cm-long loop of string around the three pins.



Place a pencil or a pen between the string and the triangle, pull it taut and move the pen back and forth, keeping the string taut. What type of curve do you expect to draw and why? Now move the pencil or pen and draw the curve. Does the result agree with your conjecture?

20. The blue point, B, is fixed on the arc of the circle with radius r . This circle is inside and touching another circle that has radius $2r$. The smaller circle turns inside and along the side of the larger circle. What is the locus of B? You may create a script for *The Geometer's Sketchpad* to animate this action.

