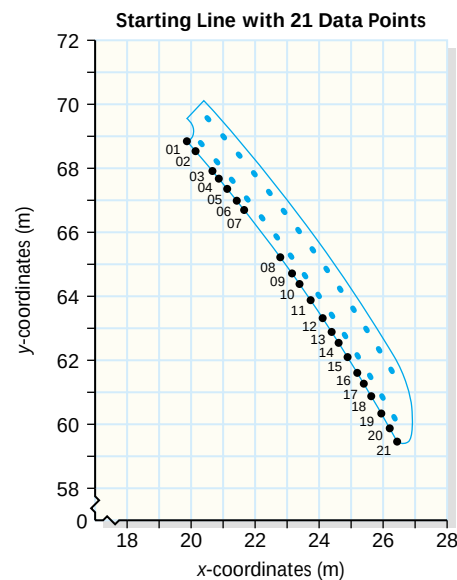


The ruins of ancient Corinth include a stadium with several courses for foot races. In 1980, archaeologists found and excavated the starting line for a racecourse dating from about 500 B.C. The starting line is the arc of a circle, about 12 m long and between 1.25 m and 1.30 m wide.

Toe grooves cut into the blocks mark the 12 starting positions for the runners. These grooves provided the researchers with data points to determine the radius and centre of the circle. This led to insights about the design of racecourses in ancient Greece and about the units for length and angle used by the ancient Greeks. The archaeologists knew that they could use three points on the circle to find the equation of the circle. They used several sets of three points to verify their findings.

To find the equation of a circle, you will need the radius and the centre of the circle.



Part 1: Developing the Equation of a Circle

When you draw a circle, you most likely follow these steps:

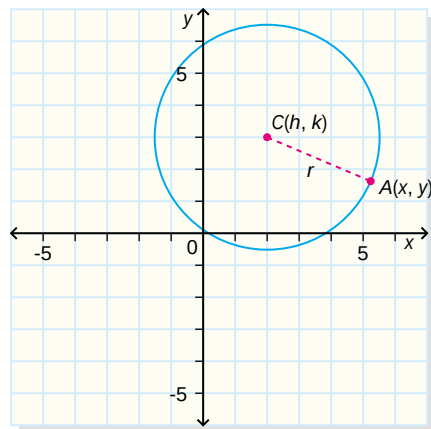
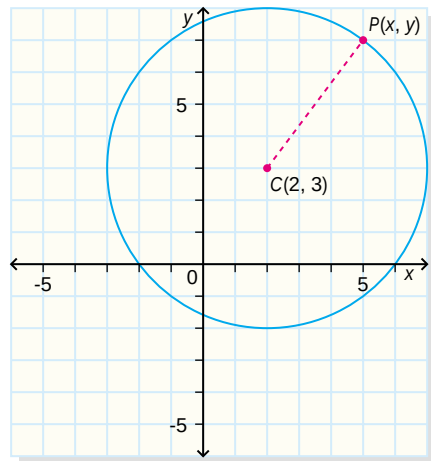
- Choose a point to be the centre.
- Use compasses or a string to measure the radius.
- Draw a curve, which represents all the points that are a given distance from the centre of the circle.

A circle is a locus of points that are equidistant from one other point, the centre. In grade 10, the centres of most circles that you encountered were at the origin. The equation for a circle with its centre at the origin is $x^2 + y^2 = r^2$. In this section, you will determine the equation of a circle with a centre that is not $(0, 0)$.

Think, Do, Discuss

The centre of a circle is $C(2, 3)$ and the circle's radius is 5 units.

1. To find the equation of this circle, begin by recalling the distance formula. The distance from $C(2, 3)$ to $P(x, y)$ is 5 units. Substitute this information in the distance formula.
2. The distance formula has a radical sign. Take steps to “remove” this radical sign. You now have an equation for the circle that is centred at $(2, 3)$ with a radius of 5 units.
3. What is the equation of the circle whose centre is $(3, -2)$ and whose radius is 4 units?
4. You can also create a general form for the equation of any circle. Describe the locus of points that are r units from the point $C(h, k)$. Let a point on the circle be $A(x, y)$. Write the formula for the distance from C to A .
5. Since you know that the distance CA is the radius, substitute r for CA in the formula in step 4. You now have the general equation for a circle with centre (h, k) and radius r .
6. What are the centre and radius of the circle with equation $(x - 1)^2 + (y + 2)^2 = 25$?
7. What is the equation of the circle with centre $(0, 3)$ and radius 8 units?
8. The equations thus far are in standard form. Expand the equation in step 6 to create an equation in the general form $x^2 + y^2 + 2gx + 2fy + C = 0$.
9. What are the values of g and f for this equation? What do these values tell you?
10. How would you change an equation in general form to standard form? Change the equation $x^2 + y^2 - 2x - 15 = 0$ to standard form and state the centre and the radius. Use what you learned in grade 10 about completing the square.

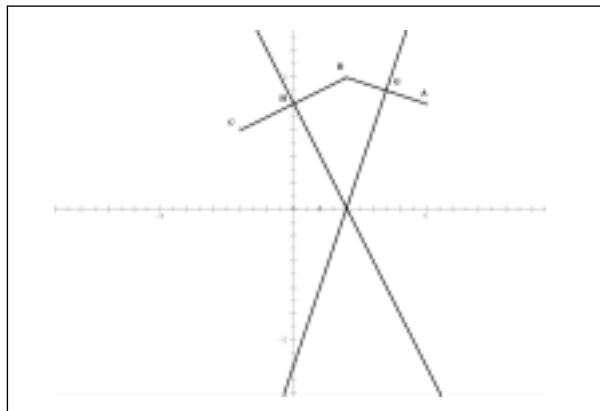


Part 2: Using Three Points to Find the Equation of a Circle

Think, Do, Discuss

1. You know that three points on a circle are $A(5, 4)$, $B(2, 5)$, and $C(-2, 3)$. How could you find the centre and radius of the circle?

- You also know that the centre of the circle must be equidistant from all three points. Take two of the points, such as A and B , and describe the locus of points that are equidistant from points A and B , using geometry and by writing an equation.



- Describe the locus of points that are equidistant from points B and C , using geometry or by writing an equation.
- You can continue to solve this problem geometrically or algebraically. For an algebraic solution, combine the equations that you found in steps 2 and 3. Then find the centre, using geometry or a graph. (The graph shown here was created with *The Geometer's Sketchpad*.)
- Find the radius of the circle.

Focus 7.2U

Key Ideas

- A circle is the set or locus of all points in a plane that are equidistant from a fixed point. The fixed point is called the centre of the circle.

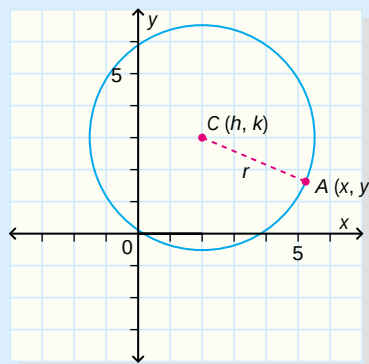
- The equation for a circle with centre $(0, 0)$, the origin, and radius r units is $x^2 + y^2 = r^2$

- The equation in standard form for a circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$

- The general form of the equation of a circle is $x^2 + y^2 + 2gx + 2fy + C = 0$

This equation can be expressed in standard form by completing the square.

- A circle has an infinite number of lines of symmetry, all passing through the centre of the circle.



Example 1

- Find the equation of the circle tangent to the x -axis and with centre $(3, 6)$.
- Write the equation of the locus of points that are 8 units from the point $(2, -3)$.

Solution

- (a) The equation of the circle in standard form is $(x - h)^2 + (y - k)^2 = r^2$.

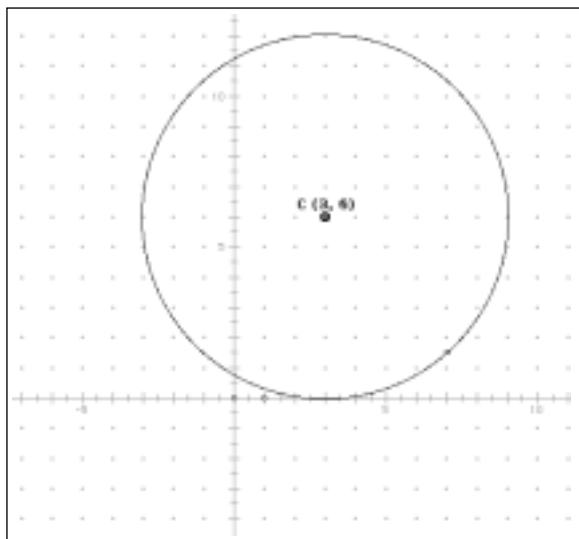
Since the centre is $(3, 6)$, the equation is $(x - 3)^2 + (y - 6)^2 = r^2$.

To find r , consider that the circle is tangent to the x -axis, as shown in the *The Geometer's Sketchpad* diagram.

The radius is then 6 units, since the centre is 6 units from the x -axis.

The equation is
 $(x - 3)^2 + (y - 6)^2 = 36$.

- (b) All of the points that are 8 units from the point $(2, -3)$ form a circle with radius 8 units and centre $(2, -3)$.
 The equation of the circle is $(x - 2)^2 + (y + 3)^2 = 64$.



Example 2

Find the centre and the radius of the circle $x^2 + y^2 - 10x + 4y + 17 = 0$.

Solution

Write the equation in standard form by completing the square.

$$x^2 - 10x + y^2 + 4y = -17$$

$$x^2 - 10x + ? + y^2 + 4y + ? = -17 + ? + ?$$

Rearrange the equation.

Determine the value that must be added to create each perfect square. Add these two values to both sides of the equation.

$$x^2 - 10x + 25 + y^2 + 4y + 4 = -17 + 25 + 4 \quad \text{Factor and simplify.}$$

$$(x - 5)^2 + (y + 2)^2 = 12$$

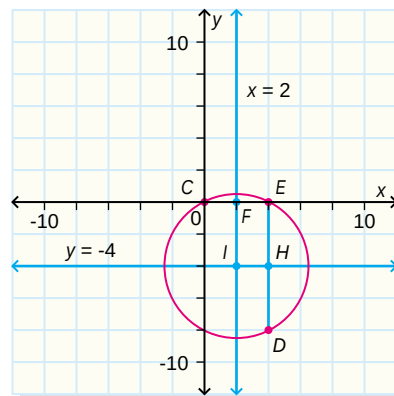
The centre of the circle is $(5, -2)$ and the radius is $\sqrt{12}$.

Example 3

Find the equation of the circle that passes through $C(0, 0)$, $D(4, -8)$, and $E(4, 0)$.

Solution

Draw a diagram. The perpendicular bisectors of CE and ED have the equations $x = 2$ and $y = -4$, respectively. The intersection point, $(2, -4)$, of the perpendicular bisectors is the centre of the circle. To find the radius, find the distance from $(2, -4)$ to any of the given points. Let one point be $(0, 0)$.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In this case, d is the radius of the circle.

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$r = \sqrt{(2 - 0)^2 + (-4 - 0)^2}$$

$$r = \sqrt{20}$$

The equation of the circle is $(x - 2)^2 + (y + 4)^2 = 20$.

Practise, Apply, Solve 7.2U

A

1. Write the equation of the circle with
 - (a) centre (0, 2) and radius 4 units
 - (b) centre (3, 0) and radius 1 unit
 - (c) centre (1, 5) and radius 9 units
 - (d) centre (-2, 4) and radius 8 units
 - (e) centre (-2, -5) and radius $\sqrt{6}$ units
 - (f) centre (4, -6) and radius $2\sqrt{5}$ units
2. Find the centre and radius of each of the following circles.
 - (a) $x^2 + y^2 = 18$
 - (b) $(x - 2)^2 + y^2 = 9$
 - (c) $(x - 6)^2 + (y + 8)^2 = 81$
 - (d) $(x + 4)^2 + (y - 3)^2 = 32$
 - (e) $(x - 5)^2 + (y - 4)^2 = 27$
 - (f) $(x - 1)^2 + (y + 6)^2 = 40$
3.
 - i. Apply the given transformations to each circle and determine the equation of the new circle.
 - ii. On the same axes, sketch the original circle and the new circle.
 - (a) $x^2 + y^2 = 9$, translated 3 units to the right and 2 units up
 - (b) $x^2 + y^2 = 4$, translated 4 units to the left and 3 units down
 - (c) $x^2 + y^2 = 1$, translated 4 units to the right and 4 units down
 - (d) $(x - 2)^2 + (y - 3)^2 = 9$, translated 2 units to the left and 3 units down
 - (e) $(x + 4)^2 + (y - 5)^2 = 9$, translated 6 units to the right and 3 units down
4. Express each equation in general form.
 - (a) $x^2 + y^2 = 25$
 - (b) $(x - 1)^2 + (y - 4)^2 = 16$
 - (c) $(x + 4)^2 + (y - 5)^2 = 1$
 - (d) $(x + 2)^2 + (y - 3)^2 = 9$
5. Express each equation in standard form.
 - (a) $x^2 + 8x + y^2 + 6y = 0$
 - (b) $x^2 + y^2 - 4x + 2y - 11 = 0$
 - (c) $x^2 + y^2 - 2x + 10y = -22$
 - (d) $x^2 + y^2 - 12x + 6y = 4$

6. Knowledge and Understanding: Find the radius and centre of each circle.

- (a) $x^2 + y^2 - 4y = 0$
- (b) $4x^2 + 4y^2 = 16$
- (c) $2x^2 + 2y^2 - 6x + 4y + 1 = 0$
- (d) $x^2 + y^2 - 6x + 2y + 1 = 0$
- (e) $x^2 + y^2 - 8x + 6y = 11$

7. Find an equation for the locus such that

- (a) all points are 2 units from the origin
- (b) all points are 3 units from $(-1, 2)$
- (c) all points are $\sqrt{5}$ units from $(0, 4)$
- (d) all points are $2\sqrt{6}$ units from $(-2, 7)$

B

8. Find the equation of the circle that is tangent to the x -axis and has centre $(-3, 4)$.

9. Communication: Write the equations of the family of circles whose centres have the same x - and y -coordinates. The radius of each circle is 8 units. Describe this set of circles.

10. Application: Find the equation of each circle.

- (a) The diameter has end points $A(-2, 4)$ and $B(2, 8)$.
- (b) The diameter has end points $A(1, 2)$ and $B(3, -6)$.
- (c) The centre is $(0, 0)$ and the x -intercept is 8.
- (d) The centre is $(0, 2)$ and the y -intercept is -2 .
- (e) The centre is $(-1, 2)$ and the x -intercept is 3.

11. Is $x^2 + y^2 - 8x + 6y + 30 = 0$ the equation of a circle? Justify your answer.

12. Graph each equation. (The graph of each equation is a semicircle.)

- | | |
|------------------------------|-------------------------------|
| (a) $y = \sqrt{4 - x^2}$ | (b) $y = \sqrt{9 - x^2}$ |
| (c) $y = -\sqrt{16 - x^2}$ | (d) $x = \sqrt{9 - y^2}$ |
| (e) $y = 2 + \sqrt{9 - x^2}$ | (f) $y = 3 - \sqrt{25 - x^2}$ |

13. Find the equation of each of the following circles.

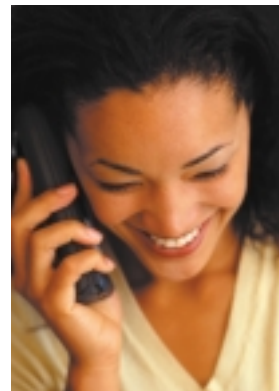
- (a) centre $(0, 0)$ and passing through $(3, 4)$
- (b) centre $(2, 3)$ and passing through $(4, 7)$
- (c) centre $(0, 0)$ and passing through $(1, 2)$
- (d) centre $(-1, 0)$ and passing through $(3, 1)$
- (e) centre $(8, 2)$ and passing through $(1, 4)$

14. The centre of a circle is $(1, 2)$ and the radius is 5 units. For this circle, find the length of a chord that is 4 units above the centre and that is parallel to the x -axis.

- 15. Check Your Understanding:** The general form of the equation of a circle is $x^2 + y^2 - 8x + 6y + 13 = 0$.
- (a) Explain how to change this equation to standard form.
 - (b) Write the equation in standard form.
 - (c) What are the advantages of the standard form?

C

- 16.** A cellular phone tower is located at point $(0, 0)$. The tower emits a signal that is accessible in a range of 30 km. Another tower is 50 km northeast of the first tower, and its signal also has a range of 30 km. These two towers are the only ones in the area.
- (a) If you are at the point $(40, 20)$, can you make a call using a cell phone? Each linear unit represents 1 km.
 - (b) What is the area of the overlap of the towers' ranges?
 - (c) What is the minimum distance that you can be within the range of one tower, but not within the range of the other tower?



- 17. Thinking, Inquiry, Problem Solving:** Find the equation of all circles, each with radius 8 units, that are tangent to the graphs of both $y = x$ and $y = -x$.
- 18.** A point moves so that it is always twice as far from $(5, 0)$ as it is from $(2, 0)$. Describe the curve on which it travels and write the equation of the curve.
- 19.** Draw the semicircle $y = \sqrt{25 - x^2}$.
- (a) Locate the x -intercepts and call them A and B .
 - (b) Locate the y -intercept and label it C .
 - (c) Prove that $\triangle ABC$ is a right triangle.
- 20.** Determine the points of intersection between the circles $(x + 4)^2 + (y - 1)^2 = 25$ and $(x - 2)^2 + (y - 3)^2 = 25$. State the coordinates to the nearest hundredth.



The Chapter Problem—Designing a Bridge

In this section, you studied equations of circles. Apply what you have learned to create a semicircular arch.

- CP1.** What is the radius? Write the equation of the curve.
- CP2.** Graph the curve.
- CP3.** Determine if two trucks will be able to pass under the arch. Leave space between the trucks, as well as at each side of the road.
- CP4.** Is this a good design? Why or why not?