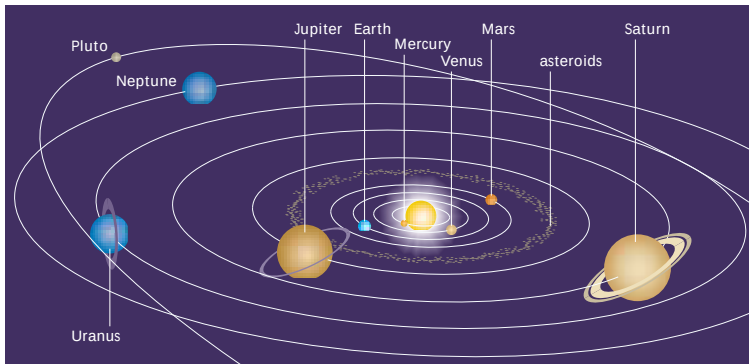


In the sixteenth century, Nicolaus Copernicus claimed that the sun, not the Earth, was the centre of the solar system. He believed that the orbits of planets were circles. A century later, Johannes Kepler (1571–1630) discovered that the orbits of the planets were ellipses or “stretched circles.” An ellipse has two fixed points, called **focal points**, or **foci**. The sun is at one of the focal points of the Earth’s orbit.



Nicolaus Copernicus (1473–1543)



Johannes Kepler (1571–1630)

Part 1: Stretching a Circle

Think, Do, Discuss

1. Graph $x^2 + 4y^2 = 16$ and describe the graph.
(Hint: Construct a table that includes the x - and y -intercepts.)
2. What is the shape of the graph? What are the x -intercepts? the y -intercepts?
Along which axis is the ellipse wider, the x -axis or the y -axis?
3. When an ellipse is centred about the origin, the line segment between the x -intercepts and the line segment between the y -intercepts are called the axes of the ellipse. The longer segment is the major axis, and the shorter axis is the minor axis. Label the major axis and the minor axis on your graph.
4. What is the length of the major axis?
5. Divide both sides of the equation in step 1 by 16. Discuss these aspects of the new equation:
 - (a) the x -intercepts and the y -intercepts
 - (b) the length of the major axis and the length of the minor axis

6. For the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, predict the x - and y -intercepts. Either draw a graph or use algebra to confirm your prediction. Where is the major axis? What is the length of the major axis?
7. The standard form of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. What are the properties of the graph of this equation?
8. What would be different about the graph of $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a > b$?
9. All of the above ellipses are centred about the origin. Recalling what you know about circles and given that an ellipse is a “stretched circle,” find the centre of the ellipse $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$.
10. It would be difficult to find the x - and y -intercepts of the ellipse in step 9, but it is not difficult to find the lengths of the major and the minor axes. Find the lengths of the major and the minor axes. Sketch a graph of the ellipse in step 9.

Part 2: The Locus Definition of an Ellipse

Another way of looking at an ellipse is to use a locus definition. The following construction of an ellipse is based on the locus definition of an ellipse.

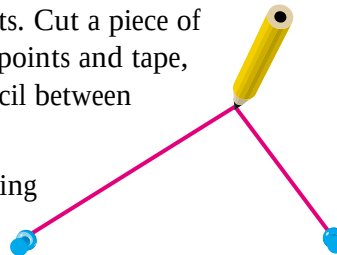
Choose two points on a sheet of paper as two fixed points. Cut a piece of string that is longer than the distance between the fixed points and tape, or tack, it to the two fixed points. Insert the tip of a pencil between the string and the fixed points, and pull the string taut.

If you move the pencil around the paper, keeping the string taut, you will draw an ellipse. Use the same length of string, but move the fixed points closer together.

Notice how the new ellipse has a different shape. Leave the fixed points where they are and change the length of the string. Again, you have created a new ellipse.

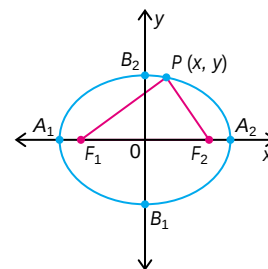
The locus definition of an ellipse:

An ellipse is the set or locus of points in a plane such that the sum of the distances from two fixed points to any other point is constant.



Think, Do, Discuss

1. Explain how the diagram on the right illustrates the locus definition of the ellipse.
2. Describe how you would construct an ellipse so that the major axis would be horizontal. How would you construct an ellipse with a vertical major axis?
3. Imagine that the ellipse in the diagram is constructed with a piece of string that is 10 units long. What is the relationship between PF_1 and PF_2 ?



4. What is the length of the major axis for any ellipse?
5. Show that the length of the string is equal to the length of the major axis by moving $P(x, y)$ to point A_2 .
6. Move $P(x, y)$ to point B_2 . What is the length of F_1B_2 ?
7. Let a be the distance from the centre of the ellipse to one of the vertices of the major axis, A_2 . (Compare this distance to one of the focal radii.) Let b be the distance from the centre to one of the intercepts of the minor axis, B_2 , and let c be the distance from the fixed point F_2 to the centre. Use your diagram to explain why $a^2 = b^2 + c^2$.
8. For the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, determine a and b and use these values to determine c . What are the coordinates of the foci for this ellipse? What is the length of string that you would need to draw this ellipse?

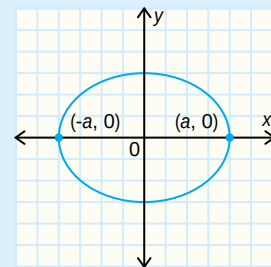
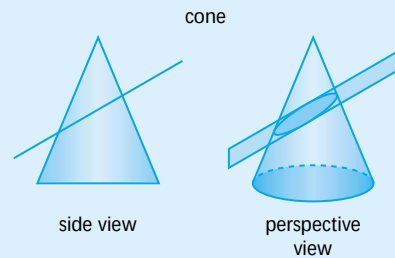
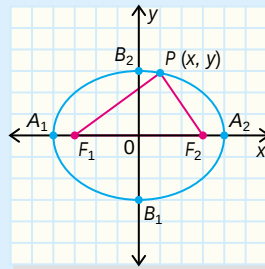
Focus 7.4U

Key Ideas

- The ellipse is created by a plane slicing through both sides of a cone at a diagonal angle.
- An ellipse is a set or locus of points in a plane such that the sum of the distances from two fixed points to any other point is constant. The sum $PF_1 + PF_2$ is a constant.
- This ellipse is centred about the origin, and its major axis lies along the x -axis.
- An ellipse has two lines of symmetry, A_1A_2 and B_1B_2 , that are perpendicular to one another. The line A_1A_2 is the major axis, the longer axis, and B_1B_2 is the minor axis.
- A_1 and A_2 are called the vertices of the ellipse and are each $|a|$ units from the centre, $(0, 0)$, of the ellipse. The length of the major axis is $2a$. The length of the minor axis is $2b$.
- The standard form of the equation for an ellipse whose centre is the origin and whose major axis is along the x -axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a > b$$

The vertices are $(a, 0)$ and $(-a, 0)$.

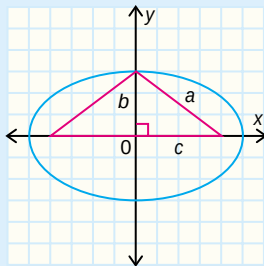
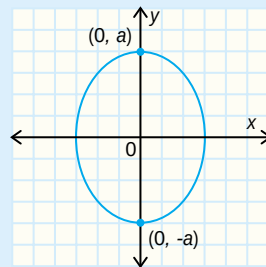


- The standard form of the equation for an ellipse whose centre is the origin and whose major axis is along the y -axis is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \text{ where } a > b$$

The vertices are $(0, a)$ and $(0, -a)$.

- The foci, or focal points, of the ellipse are F_1 and F_2 . The foci lie on the major axis and are each $|c|$ units from the centre of the ellipse.
- In any ellipse, $a^2 = b^2 + c^2$.



- The standard form of the equations for the ellipse whose centre is (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \text{ for a horizontal major axis, where } a > b, \text{ or}$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \text{ for a vertical major axis, where } a > b$$

Example 1

Find the equation of the ellipse with x -intercepts ± 4 and y -intercepts ± 3 .

Solution

Because the positive x -intercept is larger than the positive y -intercept, let the positive x -intercept be a , and let the positive y -intercept be b . The major axis lies along the x -axis and the equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Example 2

For the conic $\frac{x^2}{25} + \frac{y^2}{64} = 1$,

- what are the x -intercepts?
- what are the y -intercepts?
- state the coordinates of the foci
- find the length of the major axis
- graph the conic

Solution

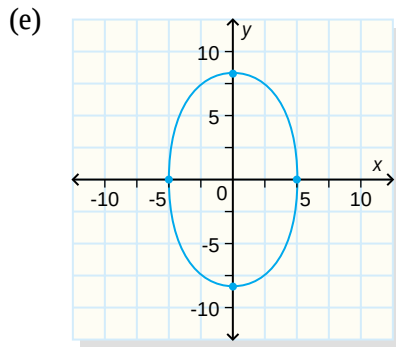
The major axis lies along the y -axis, and $b = 5$ and $a = 8$.

- (a) The x -intercepts are ± 5 .
 (b) The y -intercepts are ± 8 .
 (c) Since $a^2 = b^2 + c^2$, then $8^2 = 5^2 + c^2$.

$$\begin{aligned} 64 &= 25 + c^2 \\ c^2 &= 64 - 25 \\ c^2 &= 39 \\ \sqrt{c^2} &= \pm\sqrt{39} \\ c &= \pm\sqrt{39} \end{aligned}$$

Since the centre of the ellipse is $(0, 0)$ and the foci lie on the major axis, the y -axis, then the foci are $(0, \sqrt{39})$ and $(0, -\sqrt{39})$.

- (d) The length of the major axis is $2a$ and $a = 8$. The length is then $2(8) = 16$ units.

**Example 3**

Find the equation of the ellipse with centre $(0, 0)$, foci $(5, 0)$ and $(-5, 0)$, and y -intercepts ± 6 .

Solution

The foci are on the x -axis. Therefore, the major axis is horizontal. The equation is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, and $b = 6$ and $c = 5$. Find a .

$$\begin{aligned} a^2 &= b^2 + c^2 \\ a^2 &= 6^2 + 5^2 \\ a^2 &= 36 + 25 \\ a &= \pm\sqrt{61} \end{aligned}$$

The equation of the ellipse is

$$\frac{x^2}{61} + \frac{y^2}{36} = 1$$

Example 4

Find the centre and the length of the major and the minor axes for the ellipse $4(x - 2)^2 + 16(y + 1)^2 = 64$.

Solution

Change the form of the equation to standard form by dividing by 64.

$$\begin{aligned}\frac{4(x - 2)^2}{64} + \frac{16(y + 1)^2}{64} &= \frac{64}{64} && \text{Simplify.} \\ \frac{(x - 2)^2}{16} + \frac{(y + 1)^2}{4} &= 1\end{aligned}$$

The centre of the ellipse is $(2, -1)$, and $a = 4$ and $b = 2$.

The length of the major axis is 8 units, and the length of the minor axis is 4 units.

Example 5

The shape of an arch is a semiellipse, and its span is the major axis of the ellipse. If the span is 60 m and the height is 20 m, find the height of the arch at a point 12 m from the centre, to the nearest tenth of a metre.

Solution

Draw a diagram and centre the ellipse about the origin, with the vertices on the x -axis.

From the diagram, $a = 30$ and $b = 20$.

The equation of the ellipse is

$$\frac{x^2}{900} + \frac{y^2}{400} = 1$$

Substitute $x = 12$ in the equation.

$$\frac{12^2}{900} + \frac{y^2}{400} = 1 \quad \text{Expand.}$$

$$\frac{144}{900} + \frac{y^2}{400} = 1 \quad \text{Simplify and rearrange.}$$

$$\frac{y^2}{400} = 1 - \frac{4}{25} \quad \text{Subtract. Then multiply both sides by 400.}$$

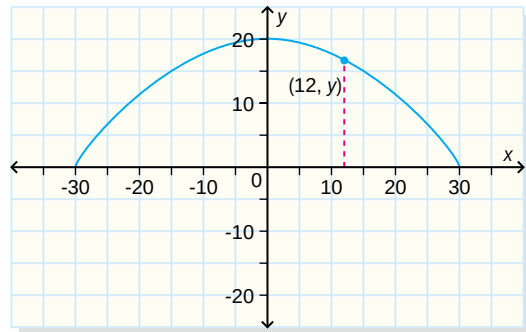
$$y^2 = 400\left(\frac{21}{25}\right) \quad \text{Simplify.}$$

$$y^2 = 16(21)$$

$$\sqrt{y^2} = \sqrt{16(21)}$$

$$y = \pm 4\sqrt{21} \quad \text{The value of } y \text{ must be positive.}$$

$$y \doteq 18.3$$



The height of the arch at a point 12 m from the centre is about 18.3 m.

A

1. For each ellipse, state the vertices and the lengths of the major and the minor axes. Then sketch the graph.

(a) $\frac{x^2}{4} + \frac{y^2}{9} = 1$	(b) $\frac{x^2}{25} + \frac{y^2}{16} = 1$
(c) $\frac{x^2}{49} + \frac{y^2}{9} = 1$	(d) $\frac{x^2}{49} + \frac{y^2}{16} = 1$
(e) $\frac{x^2}{25} + \frac{y^2}{100} = 1$	(f) $\frac{x^2}{81} + \frac{y^2}{64} = 1$
2. For each ellipse, state the centre, the lengths of the major and the minor axes, and whether the major axis is horizontal or vertical. Then sketch the graph.

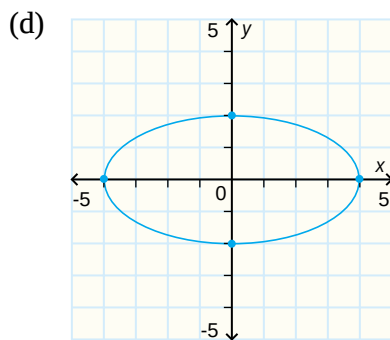
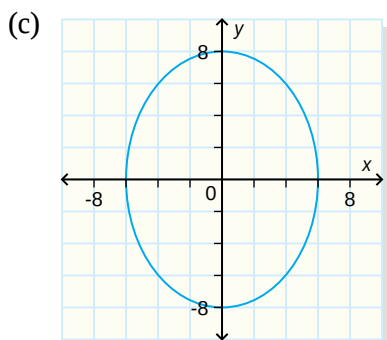
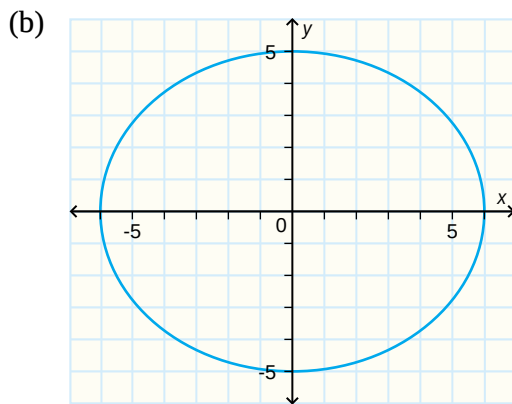
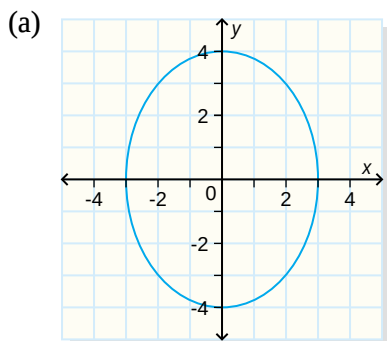
(a) $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$	(b) $\frac{(x+1)^2}{25} + (y-3)^2 = 1$
(c) $\frac{(x-3)^2}{100} + \frac{(y+4)^2}{16} = 1$	(d) $\frac{(x-1)^2}{2} + \frac{(y+2)^2}{4} = 1$
(e) $(x+1)^2 + \frac{(y+1)^2}{4} = 16$	(f) $4(x+3)^2 + 9(y-4)^2 = 36$
3. **Knowledge and Understanding:** State the lengths of the major and the minor axes, the coordinates of the foci, and sketch the graph for each of the following ellipses.

(a) $9x^2 + 4y^2 = 144$	(b) $2x^2 + y^2 = 36$
(c) $6x^2 + 4y^2 = 48$	(d) $x^2 + 8y^2 = 72$
(e) $4x^2 + 16y^2 = 64$	(f) $25x^2 + 4y^2 = 100$
4. Each ellipse is centred about the origin. Find an equation for each ellipse.
 - (a) major axis: along the y -axis and 9 units long; minor axis: 6 units long
 - (b) minor axis: 6 units long; one vertex at $(-5, 0)$
 - (c) foci: $(-8, 0)$ and $(8, 0)$; major axis: 20 units long
 - (d) one vertex at $(4, 0)$; one focus at $(-2, 0)$
 - (e) one vertex at $(0, 8)$; one focus at $(0, 6)$
5. For each ellipse, sketch the graph.

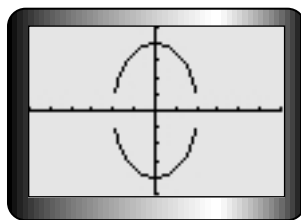
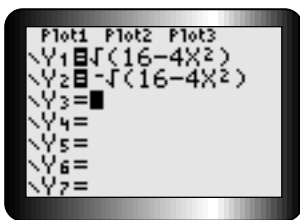
(a) $\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$	(b) $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$
(c) $2x^2 + y^2 = 36$	(d) $\frac{(x+3)^2}{100} + \frac{(y-1)^2}{49} = 1$
(e) $5x^2 + 2(y-3)^2 = 80$	(f) $(x-1)^2 + 9y^2 = 9$

B

6. For each graph, find
- the coordinates of its vertices
 - the lengths of the major and the minor axes
 - the coordinates of the foci
 - the equation of the ellipse
 - the domain and the range



7. **Communication:** An ellipse has foci $(-49, 0)$ and $(49, 0)$ and x -intercepts ± 50 . Describe the appearance of this ellipse and justify your description.
8. State the centre and the lengths of the major and the minor axes of each ellipse.
- $4(x - 2)^2 + (y + 1)^2 = 16$
 - $2(x + 1)^2 + 16(y + 2)^2 = 32$
 - $3(x + 1)^2 + 16(y - 2)^2 = 48$
 - $5x^2 + 2(y - 3)^2 = 80$
9. In the previous section, you used a graphing calculator to draw the graph of a circle. You could use the same method to draw the graph of an ellipse using a graphing calculator. To graph $4x^2 + y^2 = 16$, arrange the equation into two parts: $y = \sqrt{16 - 4x^2}$ and $y = -\sqrt{16 - 4x^2}$. Enter the expressions into Y1 and Y2, choose Zsquare from the Zoom menu, and **[GRAPH]**.



For each of the following, solve for y , then graph using technology.

(a) $2x^2 + y^2 = 32$ (b) $\frac{x^2}{5} + \frac{y^2}{9} = 1$ (c) $4(x - 1)^2 + y^2 = 16$

10. Find the equation of the ellipse with centre $(0, 0)$ and whose foci are on the x -axis. One focus is at $(5, 0)$, and $b = 12$.
11. Using only the locus definition of an ellipse, find the equation of the ellipse with foci $(0, 4)$ and $(0, -4)$. The sum of the focal radii is 10.
12. The Earth follows an elliptical path around the sun, with the sun at one of the foci. The length of the major axis of the elliptical orbit is 3×10^8 km. The focus is 2.5×10^6 km from the centre of the orbit.
 - (a) What is the length of the minor axis?
 - (b) Determine the equation of the Earth's orbit.
13. **Thinking, Inquiry, Problem Solving:** Find the equation of the ellipse that includes the points $(2, -2)$ and $(-\frac{2}{3}, \frac{10}{3})$, and is centred at the origin.
14. You want to give your dog a chance to run in the backyard, but you do not want him to run away. You place two stakes in the ground that are 4 m apart and attach the ends of an 8 m long rope to each of the sticks. Then you attach the dog's collar to a ring that slides along the rope. Describe the region in which the dog may roam. Include the dimensions of the region.
15. Use your knowledge of completing the square to rearrange the following equations in the standard form of an ellipse.
 - (a) $4x^2 + y^2 - 2y - 3 = 0$ (b) $4x^2 + 9y^2 - 8x + 18y = 23$
 - (c) $4x^2 + y^2 - 16x + 2y + 1 = 0$ (d) $16x^2 + 9y^2 + 90y + 81 = 0$
16. State the domain and range of each of the following ellipses.
 - (a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (b) $\frac{x^2}{16} + \frac{y^2}{36} = 1$
 - (c) $4x^2 + 25y^2 = 100$ (d) $16x^2 + 9y^2 = 144$
 - (e) $x^2 + 4y^2 + 4x + 8y = 8$ (f) $x^2 + 9y^2 + 2x - 18y = 26$
17. Find the equation of the ellipse with centre $(0, 0)$ and whose foci are on the x -axis. The length of the major axis is three times the length of the minor axis, and the graph passes through point $(3, 3)$.



- 18. Check Your Understanding:** An ellipse is a “stretched circle.” Could you describe a circle as an ellipse? If so, what are the foci of the circle? What are the lengths of the major and the minor axes? Would this circle satisfy the locus definition of the ellipse?



- 19. Application:** A truck driver, whose truck was carrying an extra large load, was driving through a small town when he became very hungry. He spotted a fast-food restaurant with a drive-through arch in the shape of a semiellipse. While driving under the centre of the arch, he noticed that the top of his truck just touched the arch at two points. He noted that the driveway was exactly 4 m wide at the base of the arch. His truck is 3.8 m high and 3.2 m wide. He wondered whether he should radio his “buddy” to let him know that his buddy’s truck might not fit under the arch. If his buddy’s truck is 3.5 m wide and 3 m high, then will he be able to drive it under the arch?
- 20.** The door of a house is in the shape of a semiellipse. The span is 100 cm wide, and the maximum height of the door is 250 cm. A sofa has to fit through the door. The sofa’s dimensions are 120 cm by 80 cm by 250 cm. Will the sofa fit through the door? Justify your answer.



The Chapter Problem—Designing a Bridge

In this section, you have worked with equations of ellipses. Apply what you have learned to create a semielliptical arch.

- CP5.** What are the lengths of the major and the minor axes? Write the equation of the curve.
- CP6.** Graph the curve.
- CP7.** Determine whether two trucks will be able to pass, side by side, under the arch. There should be space between the trucks and at each side of the road. Adjust the major and minor axes so your arch is workable.
- CP8.** Is this a good design?

Did You Know?

Why is a manhole cover always circular? Could it be elliptical or square? A circle has a constant width—the diameter. So, no matter in which direction the cover is turned, it will not slip through the “manhole.” You could not turn an elliptical cover in this way. Is there any other shape that has a constant width? Try experimenting with *The Geometer’s Sketchpad*. Start with an equilateral triangle, $\triangle ABC$, and beginning at point A , swing an arc that is equal to AB between points B and C . Do the same for the other two points in the triangle. Would this figure work as a manhole cover? Try the same construction with a square and a regular pentagon. Which figures work and which do not?