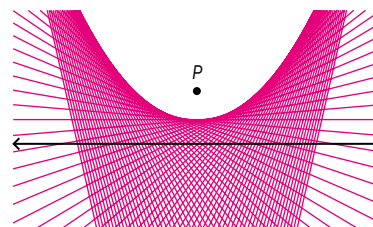


Many satellite dishes have a parabolic shape. In a parabolic reception dish, all of the rays are directed to a feedhorn, which is a fixed point or focus of the parabola. In this section, you will discover the role of the focus in the shape and equation of the parabola.



In grade 10, you learned that the graph of a quadratic equation, for example, $y = x^2$, is a parabola. In this section, you will look at a parabola in another way, using a locus definition that refers to both a focus and a fixed line, called the directrix.

In the previous section, you created a parabola using *The Geometer's Sketchpad*. This construction of the parabola featured a fixed point, or the focus, and a specific line, the directrix. When you folded the wax paper repeatedly, the fixed point, P , was the focus and the straight edge of the waxed paper was the directrix. The creation of the parabola was based on the locus definition of the parabola.



The locus definition of a parabola:

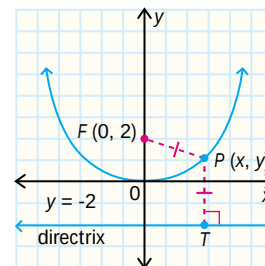
A parabola is the set or locus of points that are equidistant from a fixed point, the focus, and a fixed line, the directrix.

Part 1: Using the Locus Definition to Find the Equation of a Parabola

A parabola is the locus of all points in a plane that are the same distance from a fixed point, the focus, and a fixed line, the directrix.

Let $F(0, 2)$ be the focus and $y = -2$ be the directrix. Let $P(x, y)$ be an arbitrary point on the curve such that $PF = PT$, where PT is a perpendicular distance from the directrix to the curve.

Notice that the origin is a point on the curve since it is 2 units from both $F(0, 2)$ and the directrix.



Apply the distance formula to create an equation for the curve.

$$\begin{aligned}
 PF &= PT \\
 \sqrt{(x-0)^2 + (y-2)^2} &= \sqrt{(x-x)^2 + (y+2)^2} \\
 \sqrt{x^2 + (y-2)^2} &= \sqrt{(y+2)^2} \\
 x^2 + (y-2)^2 &= (y+2)^2 \\
 x^2 + y^2 - 4y + 4 &= y^2 + 4y + 4 \\
 x^2 &= 8y
 \end{aligned}$$

Use the distance formula.

Simplify.

Square both sides.

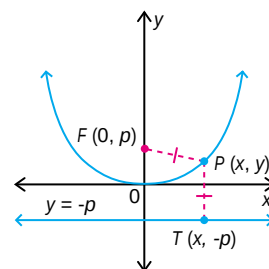
Expand.

Simplify.

The equation $x^2 = 8y$ represents the locus of points that are equidistant from point $(0, 2)$ and the line $y = -2$.

For any parabola, let $F(0, p)$ be the focus and let $y = -p$ be the directrix. Point $P(x, y)$ is a point on the parabola such that $PF = PT$.

Again notice that the origin is a point on the curve since it is equidistant from the focus and the directrix.



Apply the distance formula to create an equation for the curve.

$$\begin{aligned}
 PF &= PT \\
 \sqrt{(x-0)^2 + (y-p)^2} &= \sqrt{(x-x)^2 + (y-(-p))^2} \\
 x^2 + (y-p)^2 &= (y+p)^2 \\
 x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\
 x^2 &= 4py
 \end{aligned}$$

Use the distance formula.

Square both sides.

Expand.

Simplify.

The equation $x^2 = 4py$ describes a parabola with vertex $(0, 0)$. The focus is $F(0, p)$ and the line $y = -p$ is the directrix. Notice that $|p|$ is the focal length, the distance from the vertex to the focus.

Example 1

State the directrix and the focus of the parabola $x^2 = 8y$.

Solution

The equation $x^2 = 8y$ can be written in the form $x^2 = 4py$, with $p = 2$. The focus is $(0, 2)$ and the directrix is $y = -2$.

Example 2

State the directrix and the focus of the parabola $y^2 = 8x$.

Solution

Since y is squared rather than x , this parabola opens at the side and has the form $y^2 = 4px$. Also the focus is $(p, 0)$ and the directrix is $x = -p$. The focus of this parabola is $(2, 0)$ and the directrix is $x = -2$.

Part 2: Translating the Parabola

In Part 1, the vertex of each parabola was the origin. If the vertex is not the origin, then the equation for the parabola changes.

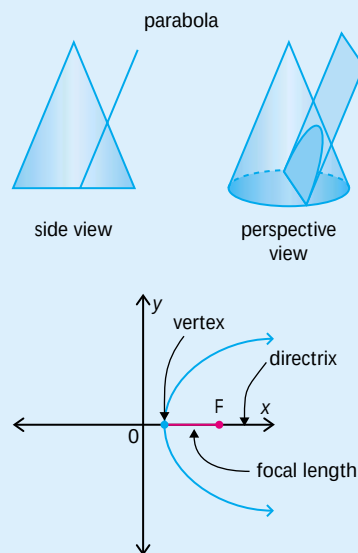
Think, Do, Discuss

1. Begin with the equation $x^2 = 4y$. Sketch this graph.
2. Using what you know about transformations, predict the shape and features of the graph of $(x - 2)^2 = 4y$. Use a sketch or graphing technology to check your prediction. Compare your sketch with the one in step 1.
3. Predict the shape and features of the graph of $(x - 2)^2 = 4(y + 1)$. Check your prediction.
4. What is the vertex of $(x - 2)^2 = 4(y + 1)$? Describe the translation of the vertex from $x^2 = 4y$ to $(x - 2)^2 = 4(y + 1)$.
5. Graph $(x - 1)^2 = y + 2$, using this new method.
6. Graph $y = (x - 1)^2 - 2$, using techniques learned in grade 10.
7. Compare and explain the results of steps 5 and 6.
8. Predict the shape and features of the graph of $(y - 1)^2 = x$. Confirm your prediction by graphing $(y - 1)^2 = x$.
9. Equations have many different forms. The standard form of the parabola is $(x - h)^2 = 4p(y - k)$. The general form of the parabola is $x^2 + 2gx + 2fy + C = 0$. Change $(x - 3)^2 = 8y$ to its general form.

Focus 7.6U

Key Ideas

- A parabola is a conic section created by a plane slicing through the side and bottom of a cone at a diagonal angle.
- A parabola is the locus of all points in a plane that are the same distance from a fixed point, called the focus, and a fixed line, called the directrix.
- The focal length of a parabola is the distance from the vertex to the focus.
- If the vertex is (h, k) , the directrix is parallel to the x -axis, and the distance from the vertex to the focus is $|p|$, then the standard form of the equation of a parabola is $(x - h)^2 = 4p(y - k)$. If p is positive, then the parabola opens up; if p is negative, then the parabola opens down.



- If the vertex is (h, k) , the directrix is parallel to the y -axis, and the distance from the vertex to the focus is $|p|$, then the standard form of the equation of a parabola is $(y - k)^2 = 4p(x - h)$. If p is positive, then the parabola opens to the right; if p is negative, then the parabola opens to the left.
- The general form of the equation of a parabola is $x^2 + 2gx + 2fy + C = 0$, if the directrix is parallel to the x -axis, and $y^2 + 2gx + 2fy + C = 0$, if the directrix is parallel to the y -axis.

Example 1

The general form of a parabola is $y^2 - 4x + 2y = 51$. Write the standard form of this conic and graph it.

Solution

To change the general form to the standard form of the conic, complete the square.

$$\begin{aligned}
 y^2 - 4x + 2y &= 51 && \text{Rearrange.} \\
 y^2 + 2y &= 51 + 4x && \text{Complete the square.} \\
 y^2 + 2y + 1 &= 51 + 4x + 1 \\
 (y + 1)^2 &= 4x + 52 \\
 (y + 1)^2 &= 4(x + 13)
 \end{aligned}$$

This parabola opens to the right, and its vertex is $(-13, -1)$.

Example 2

State the focus and the directrix of $x^2 = -5y$.

Solution

The equation is in the form $x^2 = 4py$.

The vertex of the parabola is at the origin, and $-5 = 4p$ or $p = -\frac{5}{4}$.

The focal length is $\frac{5}{4}$, the directrix is $y = \frac{5}{4}$, and the focus is $(0, -\frac{5}{4})$.

Example 3

A chain of fast-food restaurants has decided to erect a large green arch, in the shape of a parabola, outside of each of its restaurants. Each arch will be 30 m high and will be 16 m wide at its widest point. A rectangular sign will be attached to each arch so that the base of the sign is 20 m from the ground. How wide, to the nearest centimetre, must the sign be if its base just fits across the arch?

Solution

Draw a diagram and place the vertex of the parabola at the origin.

Label the graph with the given information and add information as you solve the problem. Since the parabola is symmetric, 30 m tall, and 16 m wide at the base, the two points on the curve are $(8, -30)$ and $(-8, -30)$. Then find the equation.

Since the origin is $(0, 0)$, the equation is in the form $x^2 = 4py$.

Substitute one of the points, $(8, -30)$, in the equation to find p .

$$8^2 = 4p(-30) \quad \text{Solve for } p.$$

$$64 = 4p(-30)$$

$$64 = -120p$$

$$\frac{64}{-120} = p$$

$$\frac{-8}{15} = p$$

The equation is $x^2 = \frac{-32}{15}y$.

Use the equation to determine the width of the sign.

Since the base of the sign must sit 20 m above the ground, let $(a, -10)$ represent the bottom right corner of the sign. Substitute these coordinates in the equation.

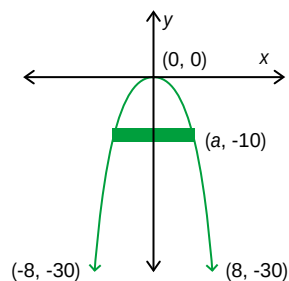
$$a^2 = \frac{-32}{15}(-10) \quad \text{Solve for } a.$$

$$a^2 = \frac{320}{15}$$

$$a = \pm \sqrt{\frac{320}{15}}$$

$$a \doteq 4.62.$$

Since a represents one-half of the width of the sign, the width of the sign is $2a$, or about 9.24 m.



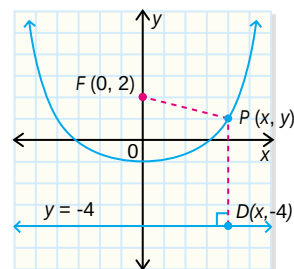
Example 4

Find the equation for the locus of points that are equidistant from point $(0, 2)$ and from the line $y = -4$. State the vertex and the focal length of the curve.

Solution

A diagram is useful. Let $P(x, y)$ be a point on the curve and $D(x, -4)$ be a point on the directrix.

The distance from the focus to $P(x, y)$ is the same as the distance from the directrix at $D(x, -4)$ to $P(x, y)$, that is, $PF = PD$. Use the distance formula to find the equation.



$PF = PD$	Use the distance formula.
$\sqrt{(x - 0)^2 + (y - 2)^2} = \sqrt{(x - x)^2 + [y - (-4)]^2}$	Simplify.
$\sqrt{x^2 + (y - 2)^2} = \sqrt{(y + 4)^2}$	Square both sides.
$x^2 + (y - 2)^2 = (y + 4)^2$	Expand.
$x^2 + y^2 - 4y + 4 = y^2 + 8y + 16$	Simplify.
$x^2 = 12y + 12$	Factor.
$x^2 = 12(y + 1)$	

The standard form for the equation of the locus is $x^2 = 12(y + 1)$. The vertex is $(0, -1)$, and the focal length is 3 units.

Practise, Apply, Solve 7.6U

A

- State the focus and the directrix of each of the following parabolas.

(a) $x^2 = -16y$	(b) $y^2 = 15x$	(c) $y = x^2$
(d) $2x^2 = y$	(e) $y^2 - 8x = 0$	(f) $10x^2 = y$
- Knowledge and Understanding:** Find the vertex and state the direction of opening for each parabola.

(a) $16(y - 4) = x^2$	(b) $y^2 = 6(x - 2)$
(c) $x = (y + 12)^2$	(d) $y = 2(x - 3)^2$
(e) $(x - 3)^2 = -12y + 24$	(f) $x + 2 = (y - 1)^2$
- The equation of each parabola is in general form. Write each equation in standard form. State the vertex and the direction of opening.

(a) $y^2 - 4x - 4y + 8 = 0$	(b) $x^2 - 4y - 8 = 0$
(c) $x^2 - 2x - 4y + 5 = 0$	(d) $y^2 - 2x + 6y + 11 = 0$
(e) $y^2 - 8x - 2y - 15 = 0$	(f) $x^2 - 4x - 2y - 16 = 0$
- Sketch each parabola in question 3.
 - State the domain and the range of each parabola in question 3.
- Communication:** Which form of the equation of the parabola, general or standard, is more useful? Support your argument.
- Find the equation of the locus of a point that moves so that it is equidistant from the line $x = 4$ and point $(-4, 0)$.
- Given the following information about each parabola, find the equation of the parabola.

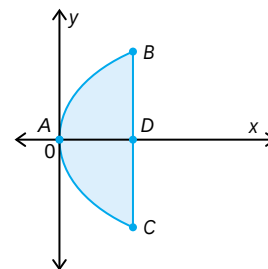
(a) focus $(3, 0)$, directrix $x = -3$	(b) focus $(0, -2)$, directrix $y = 2$
(c) vertex $(0, 0)$, directrix $x = 4$	(d) vertex $(0, 0)$, focus $(-0.5, 0)$

B

8. A parabola has a horizontal axis of symmetry and its vertex is $(0, 0)$. The curve of the parabola passes through $P(-8, -4)$.

- Find the equation of the parabola.
- What are the coordinates of the focus?
- What is the equation of the directrix?

9. The cross section of a parabolic reflector of a headlight is shown in the diagram. The distance from B to C is 32 cm and the distance from A to D is 8 cm. Determine where the bulb should be located if it is positioned at the focus.

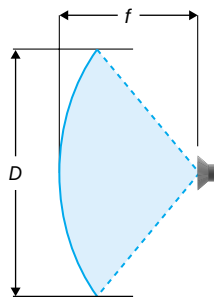


10. **Application:** Parabolic satellite dishes are deep, average, or shallow, according to the ratio of the focal length, f , to the diameter, D , of the dish at its widest part.

deep: $\frac{f}{D} < 0.3$; shallow: $\frac{f}{D} > 0.45$; average: $0.3 \leq \frac{f}{D} \leq 0.45$

Deeper dishes are preferred because they are less susceptible to interference from environmental noise. If a parabolic satellite dish is 5 m in diameter, determine the range of the focal distance for each type of dish.

- shallow
- average
- deep



11. The bulb, or focus, of a headlight's parabolic reflector is 4 cm away from the vertex of the parabola. Imagine that the vertex is at the origin. Determine the equation of the cross section of the reflector. If the depth of the reflector is 6 cm, what is the area of the circular piece of glass that covers the open end of the reflector?
12. The receiver of a satellite dish is at the focus of the parabolic dish. The focus is 80 cm from the vertex of the dish. If the dish is 4 m in diameter, find its depth.
13. To graph $(x - 2)^2 = y + 2$ using a graphing calculator, you would have to rearrange the equation only a little. Describe how you would rearrange the equation to graph $(y - 2)^2 = x + 2$ using a graphing calculator. Graph both equations using a graphing calculator.
14. Find the equation of each parabola, given the focus and the directrix.
- focus $(2, 0)$, directrix $x = -4$
 - focus $(0, 4)$, directrix $y = 1$
 - focus $(1, 2)$, directrix $y = 0$
15. Find the equation of a parabola whose vertex is $(1, 2)$ and whose vertical directrix is 2 units to the left of the focus.
16. **Check Your Understanding:** Draw a diagram of a parabola with the vertex at the origin, labelling the focus and the directrix. Then draw a diagram of a congruent parabola with vertex $(-2, 3)$. Include the focus and the directrix in your new diagram.

C

17. A parabolic arch supports a bridge that is built across a stream. The arch is 24 m wide and the height of the arch is 9 m. Find the length of the supporting vertical steel girder that is 4 m from the centre of the arch.
18. Find the equation of the parabola that passes through (5, 2), that has a vertical axis of symmetry, and that has a maximum value at (4, 3).
19. **Thinking, Inquiry, Problem Solving:** In grade 10, you learned that one form of the equation of the parabola is $y = a(x - h)^2 + k$. Now you have seen that another form is $(x - h)^2 = 4p(y - k)$.
- (a) How are these equations similar? different?
 - (b) How can you go from one form to the other?
 - (c) What are the advantages and disadvantages of each form? Support your arguments.
20. How many points of intersection are possible between two parabolas? Draw a diagram to show each case.
21. Determine the coordinates of the points of intersection between the parabolas defined by $y = x^2 - 5x - 4$ and $y = -2x^2 + 3x - 1$.



The Chapter Problem—Designing a Bridge

In this section, you have worked with equations of parabolas. Apply what you have learned to create a parabolic arch.

- CP9.** Decide where you will place the axes on the parabolic arch and then write the equation of the curve.
- CP10.** Graph the curve.
- CP11.** Determine whether two trucks will be able to pass, side by side, under the arch. There should be space between the trucks and at each side of the road. You may have to adjust some of your dimensions. If so, justify your changes.
- CP12.** Is this a good design?