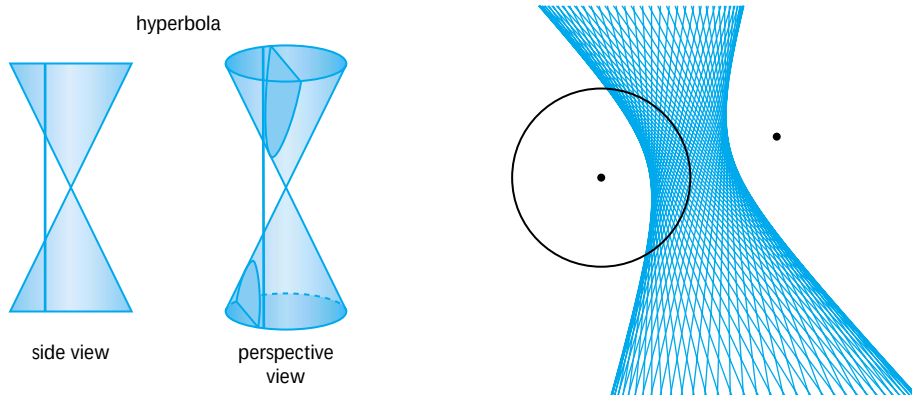


A hyperbola resembles a parabola, but it has two distinct sections that are formed by a plane slicing through the cone at a vertical angle.

You can also create a hyperbola using *The Geometer's Sketchpad*. Constructing a hyperbola is similar to constructing an ellipse (section 7.5). But, instead of starting with a point inside the circle, choose a point outside the circle.

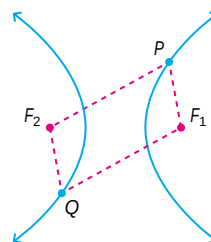


The locus definition of a hyperbola:

A hyperbola is the set or locus of points, in a plane, whose distances from two fixed points have a constant difference. The two fixed points are called the foci.

In other words, if P and Q are points on the hyperbola and F_1 and F_2 are the foci, then

$$|PF_1 - PF_2| = |QF_1 - QF_2|$$



This locus definition explains why the LORAN (Long Range Navigation) navigational system relies on hyperbolas.

Two transmitters, which are some distance apart, emit radio signals at the same time. A boat's navigator records the difference between the times when the signals reach the boat.

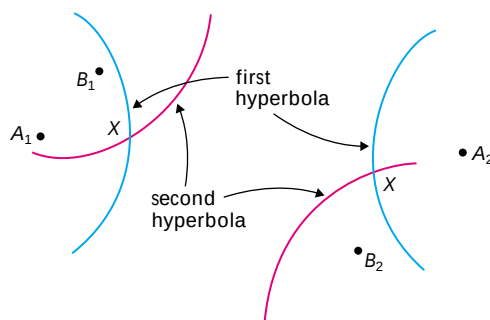
The transmitters, A_1 and A_2 , are the foci of a hyperbolic curve. The curve is the locus of all points whose distances from the foci have a constant difference, that is, the interval or the difference between the times.

The navigator refers to a map that shows a series of hyperbolic curves, called LORAN lines



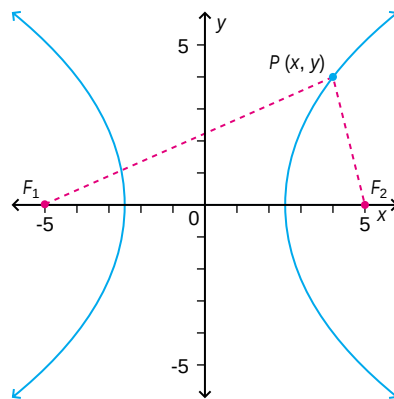
of position. The navigator finds the position of the boat on the map by matching the time interval to a specific curve on the map. This position is an estimate because there are many points on the curve, any one of which could be the boat's position.

By switching to another pair of LORAN transmitters, B_1 and B_2 , and repeating the steps, the navigator can find another hyperbola representing the boat's position. To better estimate the boat's position, the navigator locates the intersection of the two LORAN lines, X . The boat is at one of the places marked X . Determining the position of the boat in this way is similar to finding the intersection point of two curves.



Part 1: Using the Locus Definition to Find the Equation of a Hyperbola

Find the equation for the set or locus of points whose distances from two fixed points, $F_1(-5, 0)$ and $F_2(5, 0)$, have a constant difference of 6.



Let $P(x, y)$ be any point in the locus. The constant difference is 6.

$$\begin{aligned}
 |PF_1 - PF_2| &= 6 \\
 \sqrt{(x - 5)^2 + y^2} - \sqrt{(x + 5)^2 + y^2} &= 6 \\
 \sqrt{(x - 5)^2 + y^2} &= 6 + \sqrt{(x + 5)^2 + y^2} \\
 (x - 5)^2 + y^2 &= 36 + 12\sqrt{(x + 5)^2 + y^2} + (x + 5)^2 + y^2 \\
 x^2 - 10x - 25 + y^2 &= 36 + 12\sqrt{(x + 5)^2 + y^2} + x^2 + 10x + 25 + y^2 \\
 -20x - 36 &= 12\sqrt{(x + 5)^2 + y^2} \\
 -5x - 9 &= 3\sqrt{(x + 5)^2 + y^2} \\
 (-5x - 9)^2 &= 9[(x + 5)^2 + y^2] \\
 25x^2 + 90x + 81 &= 9(x^2 + 10x + 25 + y^2) \\
 25x^2 + 90x + 81 &= 9x^2 + 90x + 225 + 9y^2 \\
 16x^2 - 9y^2 &= 144 \\
 \frac{x^2}{9} - \frac{y^2}{16} &= 1
 \end{aligned}$$

Use the distance formula.

Isolate a radical.

Square both sides.

Expand.

Isolate the radical.

Divide both sides by 4.

Square both sides.

Expand and simplify.

Simplify.

Think, Do, Discuss

1. This equation, $\frac{x^2}{9} - \frac{y^2}{16} = 1$, looks similar to the equation of an ellipse. Compare this equation to the equation of an ellipse. What is different and what is the same?
2. Determine both the x - and y -intercepts of $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Draw the graph by finding six other points. You also could use graphing technology, such as *Zap-a-Graph*, or rearrange the equation so that you can enter it into a graphing calculator.
3. The foci of this hyperbola are $(-5, 0)$ and $(5, 0)$. Then $c = 5$. If $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, state the values of a and b .
4. Determine a relationship between a , b , and c .
5. The segment between the vertices is called the transverse axis. Find the length of the transverse axis.
6. Consider the other line of symmetry. The conjugate axis is perpendicular to the transverse axis and passes through the centre of the hyperbola. This axis is the segment between $(0, -b)$ and $(0, b)$. What is its length?
7. Predict what would change if you were to graph $\frac{x^2}{9} - \frac{y^2}{16} = -1$. What would be the x - or y -intercepts? Draw the graph to confirm your prediction. In this case, the equation is in the form $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$. State the values of a , b , and c and state the foci of this hyperbola.
8. Create a table or use graphing technology to graph $\frac{x^2}{4} - \frac{y^2}{36} = 1$ and $\frac{x^2}{25} - \frac{y^2}{4} = -1$. Describe the graphs, stating the x - and y -intercepts.
9. Using your knowledge of transformations, describe the graph of $\frac{(x-1)^2}{4} - \frac{y^2}{36} = 1$, without graphing the equation. Verify your answer with technology.
10. Describe the graph of $\frac{(x+2)^2}{25} - \frac{(y-1)^2}{4} = 1$. Verify your answer with technology.

Part 2: Giving Shape to a Hyperbola

It is often difficult to adequately determine the shape of a specific hyperbola if you know just a few points. But it is possible to draw an accurate graph. To begin with, you can easily locate the vertices. It would also be helpful to draw the defining “boundary” lines that are called *asymptotes*. The asymptote of a hyperbola is a line that the curve approaches, but never crosses, as x or y grow larger.

Example 1

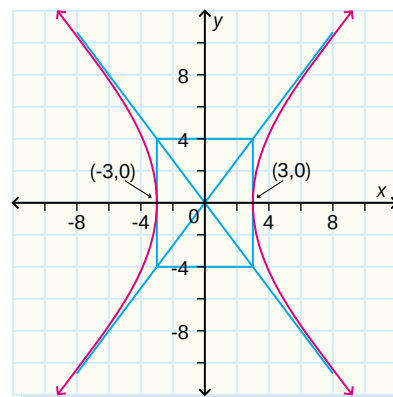
Determine the asymptotes of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

Solution

Recall that the values of a and b are 3 and 4.

To draw a graph of this hyperbola, mark the values of a and b , as shown, on the horizontal and vertical axes. Draw a rectangle, as shown. Draw and extend the diagonals of this rectangle. The extended diagonals are the asymptotes of the hyperbola.

Draw the curves of the hyperbola so that they pass through the vertices and gradually become close to, but do not touch, the asymptotes.



To determine the equations of the asymptotes, first find their slopes.

The slopes are $\frac{4}{3}$ and $-\frac{4}{3}$. Since the asymptotes pass through the origin, the y -intercepts are both $(0, 0)$. The equations of the asymptotes are $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$.

Notice that point $(3, 4)$ is on the asymptote. If you substitute $(3, 4)$ in the equation of the hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$, you will see that this point cannot be on the hyperbola. Why?

Consolidate Your Understanding

- Given $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, determine the general equations of the asymptotes by drawing a rectangle based on the values of a and b .

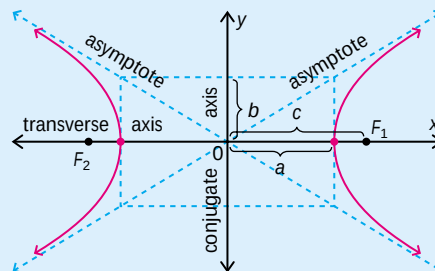
Focus 7.9U

Key Ideas

- The standard form of the equation for a hyperbola that is centred about the origin, with its foci, intercepts, and transverse axis on the x -axis, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The vertices are $(a, 0)$ and $(-a, 0)$, and the asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.



- The standard form of the equation for a hyperbola that is centred about the origin, with its foci, intercepts, and transverse axis on the y -axis, is

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1 \text{ or } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

The vertices are $(0, a)$ and $(0, -a)$, and the asymptotes are

$$y = \frac{a}{b}x \text{ and } y = -\frac{a}{b}x$$

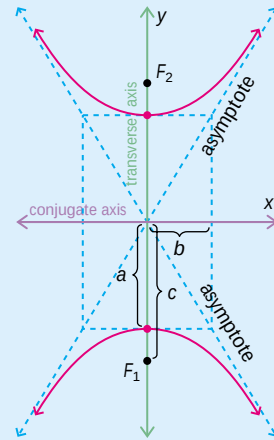
- A hyperbola has two lines of symmetry that are perpendicular to one another.
- These lines of symmetry contain the transverse axis, the end points of which are the vertices, and the conjugate axis of the hyperbola.
- If the transverse axis is along the x -axis, then the vertices are the x -intercepts of the hyperbola. If the transverse axis is along the y -axis, then the vertices are the y -intercepts.
- A hyperbola has two foci, F_1 and F_2 , that lie on the transverse axis and are each $|c|$ units from the centre.
- A hyperbola is the set or locus of all points, $P(x, y)$, in a plane, whose distances from two fixed points, F_1 and F_2 , have a constant difference: $|PF_1 - PF_2| = 2a$, the length of the transverse axis.
- The relationship between a , b , and c is

$$c^2 = a^2 + b^2$$

- A hyperbola that has been translated so that its centre is at (h, k) is in the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \text{ for a horizontal transverse axis, or}$$

$$\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = -1, \text{ for a vertical transverse axis}$$



Example 2

Find the equation of the hyperbola with foci $(-4, 0)$ and $(4, 0)$ and vertices $(-3, 0)$ and $(3, 0)$.

Solution

Since the vertices are on the x -axis, the equation of the hyperbola will be in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Since $(-4, 0)$ and $(4, 0)$ are the foci, then $c = 4$. Since $(-3, 0)$ and $(3, 0)$ are the vertices, then $a = 3$. The relationship between a , b , and c is $c^2 = a^2 + b^2$.

$$4^2 = 3^2 + b^2$$

$$16 = 9 + b^2$$

$$7 = b^2$$

$$b = \pm\sqrt{7}$$

The equation is $\frac{x^2}{9} - \frac{y^2}{7} = 1$.

Example 3

State the length of the transverse and the conjugate axes and also the vertices of the hyperbola $\frac{(x-1)^2}{4} - \frac{(y+4)^2}{16} = -1$.

Solution

Think of the equation as if the hyperbola were centred about the origin.

$$\frac{x^2}{4} - \frac{y^2}{16} = -1$$

The vertices are $(0, 4)$ and $(0, -4)$. The transverse axis is 8 units long. Since $b = 2$, the conjugate axis is 4 units long. You will see from the original equation that the hyperbola has been “translated” 1 unit right and 4 units down. This translation does not change the lengths of the transverse and the conjugate axes. The vertices, however, are translated with the hyperbola. The vertices are then $(1, -8)$ and $(1, 0)$.

Example 4

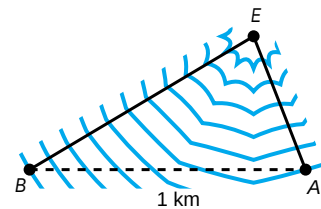
Points A and B are 1 km apart. Louise determines, because the sound of an explosion is heard at these points at different times, that the location of the explosion was 600 m closer to A than it was to B . Show that the location of the explosion is defined by a hyperbola and state its equation.

Solution

Think of points A and B as the foci of a hyperbola. Let point E be the location of the explosion.

Given: $AE - EB = 600$

Since the difference between the distances is constant, you can define a hyperbola with A and B as the foci.



$AE - EB = 2a$ and $a = 300$. Since A and B are 1000 m apart, then $c = 500$.

$$a^2 + b^2 = c^2$$

$$300^2 + b^2 = 500^2$$

$$90\,000 + b^2 = 250\,000$$

$$b^2 = 160\,000$$

$$b = 400$$

The equation is $\frac{x^2}{90\,000} - \frac{y^2}{160\,000} = 1$.

Example 5

Find the equations of the asymptotes of $4x^2 - 9y^2 = 36$.

Solution

$$4x^2 - 9y^2 = 36 \quad \text{Divide by 36.}$$

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

Then $a = 3$ and $b = 2$. Since the equations of the asymptotes are $y = \pm \frac{b}{a}x$, then the asymptotes in this case are $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$.

Practise, Apply, Solve 7.9U

A

1. Determine the intercepts of the following.

(a) $4x^2 - 9y^2 = 36$

(b) $2x^2 - 3y^2 = -12$

(c) $x^2 - y^2 = 18$

(d) $3y^2 - x^2 = 36$

(e) $6x^2 - 10y^2 = 120$

(f) $5x^2 - 10y^2 = 50$

2. Determine the values of a , b , and c for each of the following hyperbolas.

(a) $\frac{x^2}{5} - \frac{y^2}{20} = 1$

(b) $\frac{x^2}{36} - \frac{y^2}{9} = 1$

(c) $\frac{x^2}{5} - \frac{y^2}{20} = -1$

(d) $\frac{x^2}{4} - \frac{y^2}{16} = 1$

(e) $\frac{x^2}{8} - \frac{y^2}{12} = -1$

(f) $\frac{x^2}{144} - \frac{y^2}{81} = 1$

3. Sketch each of the following hyperbolas and label

i. the intercepts

ii. the foci

iii. the asymptotes

(a) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

(b) $\frac{x^2}{5} - \frac{y^2}{16} = 1$

(c) $\frac{x^2}{25} - \frac{y^2}{81} = -1$

(d) $\frac{y^2}{8} - \frac{x^2}{25} = 1$

(e) $\frac{x^2}{18} - \frac{y^2}{9} = -1$

(f) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

4. **Knowledge and Understanding:** Find the coordinates of the centre and the vertices for each of the following hyperbolas. Then graph the hyperbola, including the asymptotes in your sketch.

(a) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(b) $\frac{x^2}{25} - \frac{y^2}{4} = 1$

(c) $2x^2 - y^2 = 8$

(d) $81x^2 - 36y^2 = 2916$

(e) $\frac{(x-2)^2}{9} - \frac{y^2}{25} = 1$

(f) $(x+6)^2 - 4(y-1)^2 = 1$

5. Find the equations of the asymptotes for each hyperbola.
- (a) $x^2 - y^2 = 1$ (b) $\frac{x^2}{4} - \frac{y^2}{16} = 1$
 (c) $\frac{x^2}{9} - \frac{y^2}{25} = -1$ (d) $4x^2 - y^2 = 64$
6. For the hyperbola $\frac{x^2}{49} - \frac{y^2}{16} = 1$, sketch the graph and state the following.
- (a) the intercepts
 (b) the length of the transverse axis
 (c) the locations of the foci
 (d) the equations of the asymptotes
7. Each of the following hyperbolas is centred at the origin and satisfies the following conditions. Find the equation of each hyperbola.
- (a) The transverse axis is 12 units long, and one focal point is (0, 8).
 (b) One focal point is (0, -6), and one y -intercept is 3.
 (c) The transverse axis is 8 units long and lies along the y -axis. The conjugate axis is 4 units long.
 (d) One focal point is (5, 0), and one x -intercept is -3.
8. Sketch each graph. Comment on the similarities and differences between the graphs.
- (a) $x^2 - 9y^2 = 9$ (b) $x^2 - 9y^2 = -9$
 (c) $9x^2 - y^2 = 9$ (d) $9x^2 - y^2 = -9$
9. **Communication:** If a hyperbola is in standard form and $a = b$, the hyperbola is said to be an equilateral hyperbola. Describe and show the properties of an equilateral hyperbola.

B

10. A locus contains points, $P(x, y)$, such that the difference of the distance from point P to $A(0, 3)$ and the distance from point P to $B(0, -3)$ is 4. Find the equation of this locus and sketch the graph.
11. Sketch the graph of $\frac{(x-2)^2}{9} - \frac{(y+4)^2}{9} = -1$.
12. Given the hyperbola $\frac{x^2}{9} - \frac{y^2}{25} = 1$, find the equation of each translated hyperbola.
- (a) 3 units left and 2 units up (b) 4 units right and 3 units down
 (c) 2 units right (d) 8 units left and 4 units down
13. Create a model to determine the reflective properties of hyperbolas. Design a problem that would show these reflective properties. You may need to do some research.

- 14.** A bridge is supported by an arch in the form of the equilateral hyperbola $x^2 - y^2 = -100$. If the width at the base of the arch is 35 m, find the height of the arch at its centre.
- 15.** Write the equation of the hyperbola whose conjugate axis is 8 units long and whose vertices are $(-2, 6)$ and $(4, 6)$.
- 16.** Find the equation of the hyperbola with foci $(4, 4)$ and $(-2, 4)$. The transverse axis is 4 units long.
- 17. Application:** Two LORAN transmitters are 100 km apart. The difference between the times that the transmitter signals take to reach a ship indicates that the ship is 30 km closer to one transmitter than it is to the other. Determine the equation of the hyperbola on which the ship is located by using the positions of the transmitters as foci.
- 18. Check Your Understanding:** Show that the asymptotes of $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$ are $y = \pm \frac{a}{b}x$.
- C**
- 19. Thinking, Inquiry, Problem Solving:** Determine the equations of the asymptotes for $\frac{(x-2)^2}{9} - \frac{y^2}{25} = 1$. Then determine the equations of the asymptotes for $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
- 20.** Show that the conics $3x^2 - y^2 = 12$ and $x^2 + 3y^2 = 24$ have the same foci.

Did You Know?

Tycho Brahe (1546–1601)

Tycho Brahe was a Danish astronomer who observed the stars and planets before the telescope was invented. His eyesight was excellent, and his observations were very accurate. Brahe's observations were so accurate that Johannes Kepler was able to use them to show that planets do have elliptical orbits.

This photograph shows the Tycho Brahe Planetarium in Copenhagen, Denmark. The roof forms an ellipse, in memory of Brahe's work.

