

1.8 Compound Interest: Amount and Present Value

Specific Expectations

The following expectations are common to the Functions and Relations course and the Functions course.

- FA2.01 derive the formulas for compound interest and present value, the amount of an ordinary annuity, and the present value of an ordinary annuity, using the formulas for the n th term of a geometric sequence and the sum of the first n terms of a geometric series
- FA2.02 solve problems involving compound interest and present value
- FA2.03 solve problems involving the amount and the present value of an ordinary annuity
- FA2.05 demonstrate an understanding of the relationships between compound interest, geometric sequences, and exponential growth
- OC3.01 explain mathematical processes, methods of solution, and concepts clearly to others
- OC3.02 present problems and their solutions to a group, and answer questions about the problems and the solutions
- OC3.03 communicate solutions to problems and to findings of investigations clearly and concisely, orally and in writing, using an effective integration of essay and mathematical forms
- OC3.04 demonstrate the correct use of mathematical language, symbols, visuals (e.g., diagrams, graphs), and conventions

Evidence of Learning

Students should be able to

- identify the meaning of the variables in the compound interest formula
- identify the interest rate, the number of periods, and present or future value in verbal information
- explain the compounding of money as an exponential relation

Text Page 61

Time Required Functions and Relations or Functions: 150 min

Materials and Equipment

- spreadsheet software
- graphing calculator

Overview

- Present compound interest as an example of a geometric sequence.
- Note the exponential relation and new vocabulary: semiannually, conversion period, future value, and present value.
- Model a situation using a time line and substitute values into investment formulas.
- Use knowledge of exponents to manipulate the formula for compound interest.
- Generate tables or use a spreadsheet.

MANAGEMENT SUGGESTION

Review the commands for using a spreadsheet or provide cue cards for the necessary steps if students have not used spreadsheets recently.

Management Options

Pairs or Triplets – Students will have different levels of experience with technology. They may work with a graphing calculator, spreadsheet software, or paper and pencil.

For the *Think, Do, Discuss* questions of Part 1, give students a few minutes to work on their own and compare their results in the group before discussing these questions with the class. Answer questions about using the technology as well as about the mathematics. You may need to check on progress periodically and allow more time. Part 2 introduces new ideas and new procedures, so a Socratic approach might be appropriate for teaching options A

and B. Invite students to work through options C and D. Discuss the effect of changing only the conversion period. Students will need opportunities to practice substituting, using technology, and checking answers to develop confidence.

Teacher-Led Discussion – Write the tables on an acetate or the board. Discuss the *Think, Do, Discuss* questions, inviting students to use their calculators. Use the TI-83 view screen to show some calculations. Stop after parts 1 and 2. Assign students a few questions from the *Practice, Apply Solve*, for example, 1 to 4, so that they may confirm they understand the concepts. Then proceed with the *Think, Do, Discuss* questions of Part 3.

TECHNOLOGY OPTION

The TI-83 Plus calculator is a useful tool for calculating financial information. You will have to decide when to introduce this option, because written solutions will show students' understanding of concepts.

Part 1: Investigating Compound Interest

Think, Do, Discuss, text page 61

1. Calculate the interest earned at the end of the first year in the account opened by his parents.

$$0.05 \times \$10\,000 = \$500$$

2. Determine the balance in the account on Juan's first birthday.

$$\$10\,000 + \$500 = \$10\,500$$

3. Calculate the interest earned at the end of the second year.

$$0.05 \times \$10\,500 = \$525$$

4. Determine the balance in the account on Juan's second birthday.

$$\$10\,500 + \$525 = \$11\,025$$

5. Copy and complete the table on the next page. You could complete this table by hand, use a spreadsheet, or use the sequence editor of the TI-83 Plus calculator ($\langle 2nd \rangle \langle LIST \rangle \langle ENTER \rangle$).

Year	Balance at the Start of the Year (\$)	Interest (\$)	Balance at the End of the Year (\$)
1	10 000.00	500.00	10 500.00
2	10 500.00	525.00	11 025.00
3	11 025.00	551.25	11 576.25
...			
17	21 828.75	1091.44	22 920.19
18	22 920.19	1146.01	24 066.20

6. What is the balance in the account on Juan's 18th birthday? How much of this amount is interest?

\$24 066.20, of which \$14 066.20 is interest

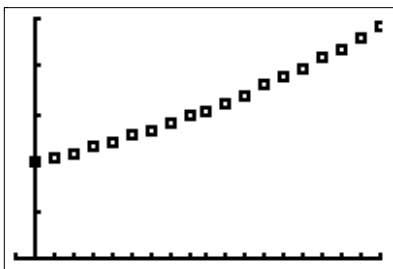
7. Write the sequence that represents the balance in the account from the 1st birthday to the 18th birthday.

10 500, 11 025, 11 576.25, 12 155.06, 12 762.82, 13 400.96, 14 071.00, 14 774.55, 15 513.28, 16 288.95, 17 103.39, 17 958.56, 18 856.49, 19 799.32, 20 789.28, 21 828.75, 22 920.19, 24066.20

8. Do the terms of this sequence increase by a constant value?

No, each year, the amount of the increase, that is, the interest, is larger than the increase of the previous year.

9. Graph this sequence.



10. Does the amount of money in the account grow at a linear or nonlinear rate? Explain.

The growth rate is nonlinear, because the graph is a curve. First differences are not constant. The amount of interest grows exponentially.

11. Using the sequence you found in step 7, divide each term by the previous term. What do you notice?

The ratio 1.05 is constant.

12. What type of sequence have you created?

The sequence is geometric, because the ratio between successive terms is constant.

13. Write each term of the sequence in step 7 as an expression of the value in step 11 and the original principal of \$10 000.

$10\,000(1.05)^1$, $10\,000(1.05)^2$,
 $10\,000(1.05)^3$, ... , $10\,000(1.05)^{18}$

14. Let n be a natural number that represents the number of years. Determine the general term of this sequence.

$$t_n = 10\,000(1.05)^n$$

15. Both Juan's parents and grandparents opened savings accounts, each with \$10 000. Explain why the balance in the compound interest account is much greater than the balance in the simple interest account on Juan's 18th birthday.

The interest in the simple interest account is always calculated on the original deposit of \$10 000. However, in the compound interest account, the interest is calculated on the yearly balance, which keeps growing, so the interest also keeps growing.

16. If Juan left the money in the compound interest account until he retired at 65, what would be the balance in the account on his 65th birthday?

$$\begin{aligned} t_{65} &= 10\,000(1.05)^{65} \\ &= 238\,399.01 \end{aligned}$$

There would be \$238 399.01 in the account.

17. Elena deposits \$5000 in an account that earns compound interest. The annual interest rate is 8%. Write the sequence that represents the yearly balance in the account each year for ten years. Determine the general term of this sequence.

The sequence is $5000(1.08)^1$, $5000(1.08)^2$, $5000(1.08)^3$, ... , $5000(1.08)^{10}$.

$$t_n = 5000(1.08)^n$$

Part 2: Defining Compound Interest—Amount

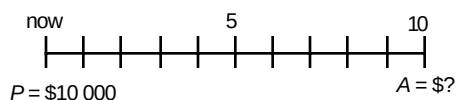
Think, Do, Discuss, text page 63

Questions 1 to 7 relate to option A.

1. How often is interest paid each year?

Draw a time line.

Interest is paid once each year.



2. How long is each conversion period?

The conversion period is one year.

3. What is the interest rate for each conversion period?

The interest rate is 12% per year.

4. How many times will interest be paid for this investment?

Interest will be paid 10 times.

5. Write the sequence that represents the balance in the account at the end of each conversion period.

11 200.00, 12 544.00, 14 049.28,
 15 735.19, 17 623.42, 19 738.23,
 22 106.81, 24 759.63, 27 730.79,
 31 058.48

6. By what factor does each new term increase? Determine the general term of this sequence.

The multiplication factor is 1.12.

$$t_n = 10\,000(1.12)^n$$

7. Determine the balance in the account at the end of ten years.

\$31 058.48

8. Repeat steps 1 to 7 for options B, C, and D.

For Option B:



Each conversion period is six months. The interest rate for each conversion period is half of 12%, or 6%. Interest will be paid twice per year for ten years, so there will be 20 payments.

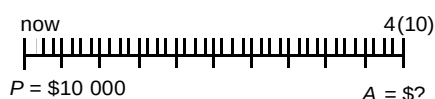
The sequence is 10 000.00, 10 600.00, 11 236.00, ... , 30 256.00, 32 071.35.

The multiplication factor is 1.06.

$$t_n = 10\,000(1.06)^n$$

The balance after ten years is \$32 071.35.

For Option C:



Each conversion period is three months. The interest rate is one-quarter of 12%, or 3%, for each conversion period. Interest will be paid 40 times in ten years.

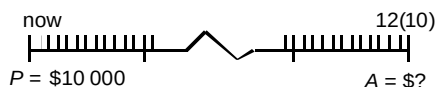
The sequence is 10 000.00, 10 300.00, 10 609.00, ... , 31 670.27, 32 620.38.

The multiplication factor is 1.03.

$$t_n = 10\,000(1.03)^n$$

The balance after ten years is \$32 620.38.

For Option D:



Each conversion period is one month. The interest rate is one-twelfth of 12%, or 1%. Interest will be paid 120 times in ten years.

The sequence is 10 100.00, 10 201.00, 10 303.01, ... , 32 677.01, 33 003.88.

The multiplication factor is 1.01.

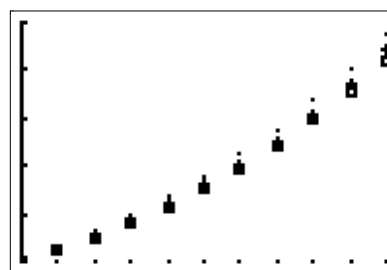
$$t_n = 10\,000(1.01)^n$$

The balance after ten years is \$33 003.88.

9. Graph each option on the same set of axes. Plot only the points that correspond to the balances in the account at the end of each year for each option.

L1	L2	L3	1
0	10000	10000	
1	11200	11236	
2	12544	12625	
3	14049	14185	
4	15735	15938	
5	17623	17908	
6	19738	20121	
L1(1)=0			

L3	L4	L5	3
10000	10000	10000	
11236	11255	11268	
12625	13167	12697	
14185	14820	14307	
15938	16680	16121	
17908	18773	18166	
20121	21129	20470	
L3(1)=10000			



Different marks in the graph will allow students to see the different rates of change. Each set of points forms a sequence, and a discussion about the common ratio for each sequence will help students understand the different growth rates.

10. Which option offers the best return on your initial investment? Explain your answer.

The shortest compounding period causes the greatest growth. When the interest is calculated monthly, the accumulated values will be largest.

Part 3: Defining Compound Interest—Present Value

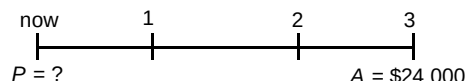
Think, Do, Discuss, text page 64

1. In this problem, which value is unknown, the future amount or the principal?

The principal, or present value, is unknown.

2. Draw a time line. Where should \$24 000 be placed on the time line?

The \$24 000 represents the value at the end of the next three years.



3. Substitute the known values in the formula $A = P(1 + i)^n$ and solve for P .

$$24\,000 = P(1 + 0.06)^3$$

$$P = 20\,150.86$$

4. What sum of money must Marcus invest today to have \$24 000 three years from now?

If Marcus invests \$20 150.86 today, it will accumulate interest and will be worth \$24 000 in three years.

5. Express the formula $A = P(1 + i)^n$ in terms of P .

$$P = \frac{A}{(1 + i)^n}$$

6. Use what you know about negative exponents to rewrite the formula for P so that n is negative.

$$P = A(1 + i)^{-n}$$

ASSESSING LEARNING SKILLS

Work Habits

Students who draw accurate time lines will more likely find accurate answers. Collect samples of assigned work to mark and allow some marks for the accurate drawing of time lines.

EXTENSION SUGGESTION

Ask students to visit a bank to get current information about interest rates and compounding periods.

Additional Example: Functions and Relations

George has x dollars, which he invests at 6%/a, compounded monthly, for y years. Investigate the effect of

- (a) doubling the original amount
- (b) doubling the period
- (c) doubling the annual interest rate
- (d) doubling the compounding period

Solution

The formula for the amount is

$$A = x \left(1 + \frac{0.06}{12} \right)^{12y}$$

- (a) $A = 2x \left(1 + \frac{0.06}{12} \right)^{12y}$, so the amount will double.

(b) $A = 2x \left(1 + \frac{0.06}{12} \right)^{2(12)y}$

The number of interest calculations will double, so growth will increase significantly. In fact, the total growth factor is squared.

(c) $A = x \left(1 + \frac{2(0.06)}{12} \right)^{12y}$

The common ratio will be significantly larger, so the investment will grow much more quickly.

$$(d) \quad A = x \left(1 + \frac{0.06}{6} \right)^{6y}$$

The amount of gain for each interest calculation is higher, but interest will be calculated fewer times.

Additional Example: Functions

Imagine that you are planning to register two years from now in a post-secondary program with an annual tuition of \$6000. What amount invested today at 5.5%/a, compounded quarterly, will pay for your first registration bill?

Solution

$$A = P(1 + i)^n$$

$$6000 = P \left(1 + \frac{0.055}{4} \right)^8$$

$$P = 5379.04$$

You will need \$5379.04.

ASSESSMENT SUGGESTION

COMMUNICATION, KNOWLEDGE, AND UNDERSTANDING

Pair students. Ask them to create a question using the following template and find the answer. Then they can trade questions, without discussion, solve, and then discuss any differences in answers or approaches.

Practise, Apply, Solve 1.8

Questions with a Functions and Relations Focus:

4–7, 11, 12, 14, 17–25

Questions with a Functions Focus:

1–7, 10–11, 15–16, 18, 20–22

6. Communication, p. 71

Management Suggestion

Pair each student with the student who sits to the left or right. Then have students answer the question. Re-pair each student with the student sitting behind the student. Students can answer the question again. Students can clarify misunderstandings by working in different pairs.

Achievement Chart Connection

Students use words and examples to communicate clearly.

Level 3 Performance

Students should recognize that the amount or future value of an investment changes as time passes, whereas the present value does not. Many different examples could be created.

ASSESSMENT SUGGESTION

KNOWLEDGE AND UNDERSTANDING

Collect and mark the responses to a specific question, for example, 4 (c).

ASSESSMENT SUGGESTION

APPLICATION

Collect and mark the responses to a specific question, for example, 10.

11. Knowledge and Understanding, p. 71

Management Suggestion

Encourage students to ask questions. Then present a similar question for students to answer without referring to their notes. Collect and mark the answers to this question.

Achievement Chart Connection

Students must demonstrate understanding of the calculation of the amount or future value of an investment.

Level 3 Performance

Students should be able to state the compound interest formula from memory, extract values from the given information, and use a calculator to solve the resulting equation.

18. Application, p. 72

Management Suggestion

Ask students to first “set up” the equations to show the substitutions before solving.

Achievement Chart Connection

Students must apply the concepts of this section to a complex situation.

Level 3 Performance

Students first calculate the future value or amount and then use this amount to calculate another future value or amount.

$$A_1 = 2000 \left(1 + \frac{0.06}{12} \right)^{60}$$

$$A_2 = A_1 \left(1 + \frac{0.08}{4} \right)^{26}$$

**21. Thinking, Inquiry, Problem Solving,
p. 72****Management Suggestion**

Allow students to work in pairs or triplets to discuss the question and plan a solution.

Achievement Chart Connection

Students must carefully plan to determine the compounding periods that will produce the same future value.

Level 3 Performance

Students should draw a time line to compare the different circumstances. The result of the first part is used in the calculations for the second part.

22. Check Your Understanding, p. 72**Management Suggestion**

Students should be able to explain their answers if they understand the effect of compounding interest. A clear justification, including calculations, indicates thorough understanding.

Ask the first person in each row or one person at each table to answer the question. Ask one person in the next row to clarify, or add to, the response. Give a third student an opportunity to add or reword this response.