

1.11 Solving Exponential Equations

Specific Expectations

Unless stated otherwise, the following expectations are common to the Functions and Relations course and the Functions course.

- FA2.02 solve problems involving compound interest and present value
- FA2.03 solve problems involving the amount and the present value of an ordinary annuity
- OC1.09(U) solve exponential equations (e.g., $4^x = 8^{x+3}$, $2^{2x} - 2^x = 12$)

Evidence of Learning

Students should be able to

- interpret a realistic financial situation by assigning variables and substituting known values
- solve an equation, or estimate a solution, when the unknown is an exponent

Text Page 88

Time Required Functions and Relations or Functions: 150 min

Materials and Equipment

- graphing calculator
- graph paper

Overview

- Graph a relation to estimate the solution to an equation.
- Solve an equation where the unknown is the exponent by expressing each side of the equation as a single power with a common base or by solving a quadratic equation.
- Solve an equation to find the number of terms in a sequence or the time involved in growth or decay.
- Solve exponential equations using zeros of the graph if graphing technology is available.

Solving Exponential Equations

Without using logarithms, students find solutions to selected exponential equations by interpolating or extrapolating from a graph, expressing each side of the equation with a common base, or guessing and checking with a calculator. This section connects exponential equations to realistic applications and sets the stage for future learning about logarithms.

MANAGEMENT SUGGESTION

Demonstrate each type of solution and allow a short practice time. Then group students and give each group an envelope in which there are pieces of paper with specific tasks written on them, for example, create a sequence, use an algebraic equation, draw a graph, and estimate a reasonable answer. Provide a common question and have students draw a task from the envelope. Allow time for work and discussion and then suggest a different question.

Management Options

Groups – Form students into groups of three or four and allow them to move through stations that address the different parts of the lesson. Some classes may respond well to the opportunity to move from one station to another in a random order, while other classes may need a more structured environment. Envelopes with a few “quick-check” questions should be available at each station. If the questions are answered correctly, the group could move to the next station. Group members should ensure that everyone in the group understands the examples before the group opens the envelope.

Teacher Lecture – If you plan to demonstrate each strategy or skill, allow students time to try some examples of each section before proceeding. Then plan for some discussion to consolidate understanding and discuss “what to do when.”

Part 1: Approximating a Solution

Think, Do, Discuss, text page 88

1. Use the formula for compound interest to write an equation for this situation.

$$A = P(1 + i)^n$$

$$4660.96 = 1000(1 + 0.08)^n$$

2. How would you describe this equation?

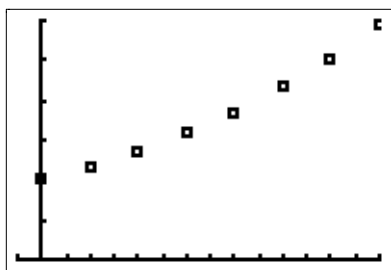
The equation is exponential because the variable is an exponent.

3. Use the expression on the right side of the equal sign to complete the table.

Years (n)	0	2	4	6
Amount (\$)	1000	1166.40	1360.49	1586.87
Years (n)	8	10	12	14
Amount (\$)	1850.93	2158.92	2518.17	2937.19

4. Using the information in the table, graph this relationship. In this case, can you draw a curve of best fit through the points? Explain.

Students should graph the data using graph paper and pencil so that they can draw the curve of best fit and also extrapolate. Their lack of experience with regression or exponential functions limits the value of the graphing calculator at this stage.



A curve of best fit could be drawn, because there are many points existing between the points already marked. Because the deposit has a value at any time, the domain of the relation is the set of real numbers. Students may be tempted to use quadratic regression, which they may have learned in grade 10. You might discuss and compare exponential and quadratic functions.

5. Can you use the graph in step 4 to solve the problem? What must you do with the graph so that you may solve the problem?

The curve of best fit must be drawn and extended to the right until the range exceeds \$4660.96.

6. Adjust the graph accordingly and determine for how long the money has been invested.

Estimates will vary depending upon the accuracy in drawing the graph, the scale selected, and the care taken in extrapolating, but students should estimate about 20 years.

7. Are you confident that your answer is accurate? Explain.

Students should share ideas about the limitations of the method.

8. Using the exponent key on your calculator, guess and check to solve the equation in step 1.

The solution to the equation is $n = 20$. Discuss how to guess and check to determine a reasonable estimate. Ensure that students have a plan, for example, making high, low, and halfway guesses, rather than just making random guesses.

9. Which method, graphing or using a calculator, gives an exact solution? Explain.

Neither method gives an accurate answer in all cases. For example, if a value for n is a fraction of a year, the estimate will be less precise. Any answer produced by graphing or using a calculator in this way is an estimate.

Part 2: Exponential Equations with the Same Base

An Algebraic Solution

TEACHER INTERVENTION

Students whose numeric skills are weak may not see where it is possible to express a number as a power, so give these students a moment to first generate several terms of a sequence and then express each term as a power.

For example, 3, 9, 27, 81, 243, 729, 2187, 6561, ... equals $3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8 \dots$

Emphasize that each side of the equation must be expressed as a single power and that the bases must be the same.

If the powers are equal and the bases are identical, the exponents must be equal. Encourage students to express this statement in their own words.

EXTENSION SUGGESTION

Present the equation $x^a = y^a$ to students and ask them to write a conjecture about the contexts in which they might find it.

Solving Exponential Equations Involving Quadratic Equations

TEACHER INTERVENTION

Review the form of a quadratic equation, referring to a diagram. Use an acetate of this diagram to show that each example is similar.

ASSESSING LEARNING SKILLS

WORK HABITS AND HOMEWORK

Because the teaching of this material will require more than one class, you might assign specific questions on day 1 and then collect and assess the responses at the beginning of class on day 2. Alternatively, other students in class might mark answers to the assigned work. Show the answers using an overhead.

EXTENSION SUGGESTION

Ask pairs of students to find information about the half-life of certain chemicals or other materials and to write a paragraph about the chemical and issues concerning its use, for example, disposal of waste materials, safety, and so on.

Finding the Number of Terms in a Geometric Sequence

In Example 3, students will see how to solve an exponential equation that is a general term for a geometric sequence. Students should see that they can find the number of terms in a sequence by solving an exponential equation using one of these techniques.

Exponential Growth and Decay Problems

Encourage students to recognize that financial questions and growth and decay questions involve identical mathematical concepts.

$$\text{New Value} = \text{Original Value} \times \text{Change Factor}^{\# \text{ of periods}}$$

Students can create and solve similar equations for an unknown quantity using the techniques of this chapter. Suggest that students find links to other courses or interests—science, archaeology, and business.

Part 3: Solving an Exponential Equation Using Graphing Technology

Sometimes the terms in an equation cannot be expressed with a common base. Without using logarithms, students can approximate a solution using a graphing calculator. The method in this example illustrates the meaning of exponents.

You might need to remind students of the connection between the zero(s) of a function and the x -intercepts of the function when $y = 0$.

Consolidate Your Understanding

1. Write an example of an exponential equation. Explain how an exponential equation is different from **(a)** a linear equation and **(b)** a quadratic equation.

Responses will vary, but students should refer to exponents and use correct mathematical terms.

2. What must be true of the expression on both sides of an exponential equation to solve it algebraically?

It must be possible to express each side of the equation as a single power using the same base or it must be possible to express the equation in quadratic form.

Students' responses should not refer to logarithms.

3. How can you verify your solutions for an exponential equation?

Substitute the answer into both sides of the original equation and solve each side. The solutions are correct if the sides are equal.

Graph the equation and identify the intersection point.

4. Do all exponential equations have exact solutions? Explain.

Some exponential equations have exact solutions, but the solutions to others are only estimates. Other solutions are irrational or unreal numbers.

Additional Example: Functions and Relations

Each student should select a unique % annual growth rate and express the corresponding change rate $(1 + \blacksquare)$.

Task: Find a way to quickly generate the time required for any amount to double; triple, quadruple; Explain why the original amount need not be specified.

Solution

The situation can be described with an exponential equation: $ka = a(1 + \blacksquare)^t$. Each student might create a graph of the function $k = (1 + \blacksquare)^t$ and read answers by interpolation. The original amount can be factored from both sides of the equation and is irrelevant for the solution.

Additional Example: Functions

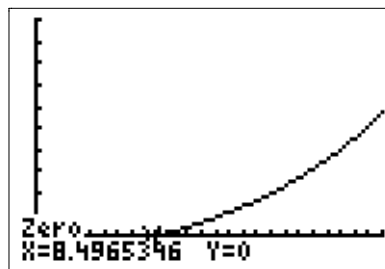
Sally is an investment counsellor who has been asked to help a client double his investment. The annual rates for different investments vary from 6.0%/a to 15.75%/a. Create four different scenarios and prepare a presentation for the client. Is the amount invested relevant?

Solution

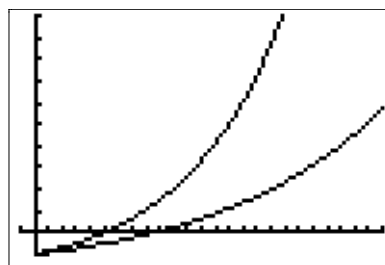
$$2a = a(1 + i)^t$$

Students will select different interest rates. Graph $y = (1 + i)^t - 2$ for each rate to identify the zero. (See the screens below.) The amount invested is not relevant, because the ratio of the new amount to the original investment is independent.

Select 8.5%.



Select 8.5% and 15%.



By graphing different scenarios on the same axes, the change in zero value, that is, the time in which the investment will double, can be seen.

Practise, Apply, Solve 1.11

Questions with a Functions and Relations Focus:

3–5, 8–11, 13, 16–24

Questions with a Functions Focus:

94–96: 1–13, 15–16, 18, 20, 22, 25

13. Knowledge and Understanding, p. 95

Management Suggestion

Give students a specific time in which to answer this question. Then collect and mark the solutions, or place a model solution on the overhead for self-assessment and do not record marks.

Achievement Chart Connection

Students should be able to solve this question without using technology.

Level 3 Performance

Students might

- express each number, 9, 81, and 27, as a power of 3 and apply exponent laws to create a single power on the right side of the equation
- make the bases equal, write an equation for the exponents, and solve the equation
- check the solution and write a final statement

16. Communication, p. 95**Management Suggestion**

Invite students to share their responses with classmates using the board, poster paper, or overhead.

Achievement Chart Connection

Students could use a graph or a diagram, algebraic equations, or written explanations demonstrating their grasp of mathematical language.

Level 3 Performance

Students might

- illustrate the solution of an exponential equation using algebra or a graph
- clearly explain their choice of method
- select non-trivial examples

18. Thinking, Inquiry, Problem Solving, p. 95**Management Suggestion**

Allow students to make some guesses about the half-life of carbon-14 before beginning researching and solving the question.

Achievement Chart Connection

Students are required to create an algebraic model of the decay process and use it to make a prediction.

Level 3 Performance

Students should

- determine an accurate value for the half-life of carbon-14
- create an appropriate exponential equation
- estimate a reasonable age for the leaf fossil

20. Application, p. 96**Management Suggestion**

Initiate a discussion about starting salaries and reasonable expectations for annual increases.

Achievement Chart Connection

Students should recognize that the relationship can be modelled using a geometric sequence or an exponential equation.

Level 3 Performance

Students should

- determine a rate of 1.06
- create an appropriate sequence and/or equation
- find a realistic solution to the question

22. Check Your Understanding, p. 95**Management Suggestion**

Students will have to first divide both sides of the equation by 2 and identify the connection between the algebraic and graphic solutions. Students might work in pairs with each partner choosing a different method.

ASSESSMENT SUGGESTION

Ask students to identify or create the match between a graph, an equation, and a description.

The Chapter Problem – Controlling Non-Native Plant Populations

By the end of this section, students should be able to organize the known information, search for and describe patterns that exist, use the patterns to find required values. Many students will have completed the problem at the end of this section.

The answers to the Chapter Problem are on p. 1-54 of the *Teacher Resource*.