

1.1 Exploring Patterns and Sequences

Specific Expectations

Unless stated otherwise, the following expectations are common to the Functions and Relations course and the Functions course.

- FA1.01(U) write terms of a sequence, given the formula for the n th term or given a recursion formula
- FA1.02(U/C) determine a formula for the n th term of a given sequence (e.g., the n th term of the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$ is $\frac{n}{n+1}$)
- FA1.04 determine the value of any term in an arithmetic or a geometric sequence, using the formula for the n th term of the sequence

Evidence of Learning

Students should be able to

- generate a table of values to represent several consecutive examples of a situation
- describe a pattern in a sequence of numbers
- describe the terms of a sequence by using a general term where the number of the term is the variable
- use a general term to generate a specific term of an arithmetic sequence

Text Page 16

Time Required Functions and Relations or Functions: 75 min

Materials and Equipment

- coins (so students can generate their own data for part D of the *Think, Do, Discuss*)
- scientific calculator

Overview

- Identify patterns in real-life situations and predict values for the dependent variable according to the number of the term.
- These words associated with sequences are introduced: general term, finite, infinite, and discrete.
- A term in the sequence can be predicted by its position in the sequence or by its relationship to the previous term. Students may confuse the two ways of describing terms; only in the Functions and Relations course are recursive descriptions required.

TECHNOLOGY SUGGESTION

If computers are available and time permits, students might explore related material via the Internet. A search for “Sierpinski,” “The Geometer’s Sketchpad,” or “TI-83” will produce many links to Web sites. To narrow the search, direct students to specific Web sites.

Terms of Sequences

When students generate and discuss several simple sequences with readily identifiable patterns, they will recognize that they can generate a specific term using an expression in which the position of the term is the independent variable. The expression, that is, the general term t_n , describes any term in the sequence. Students may not intuitively describe patterns in this way, instead, they may focus on how the numbers change and suggest a recursive version of the general term. This approach is valid, but restrict the discussion to descriptions that predict any term according to its position in the sequence, not according to its relationship to a previous term.

Students can categorize a sequence as finite or infinite by noting whether a final term exists or not.

Management Options

Encourage U/C students to consider sequences in a variety of different contexts—business, sociology, environmental studies, crime statistics, and so on.

Expect Functions and Relations students to generalize algebraically easily and to suggest alternatives to, and simplified versions of, general terms.

Functions and Relations students will move quickly to describing a sequence algebraically.

Pairs – Because students have experience in creating tables, they can work together quickly to provide data for each situation. Encourage pairs to think about the questions and to exchange ideas before confirming ideas in a class discussion. When students understand the concepts, suggest they try several examples on their own, but be supportive if they need further explanation.

Teacher-Led Demonstration – You could walk the class through this *Think, Do, Discuss* with tables on the board or overhead, or with prepared worksheets. Students could share the task of collecting data and then display the results in the class. During the class, students should progress to working individually with opportunity for clarification, if necessary.

ASSESSING LEARNING SKILLS

Initiative

To assess initiative in collecting data and in discussion, solicit ideas from all students in a class discussion and monitor the progress of student pairs.

Think, Do, Discuss, text page 17

1. Copy and complete each table. List the five numbers that you enter for each table separately.

A. Harry sells vacuum cleaners and earns \$400 each month. For each vacuum cleaner he sells, he earns an additional \$110.

Number of Vacuum Cleaners Sold	1	2	3	4	5
Total Earnings (\$)	510	620	730	840	950

B. Marka deposits \$500 only once in a savings account that pays 5% interest at the end of each year.

At End of Each Year	1	2	3	4	5
Money in Savings Account (\$)	525	551.25	578.81	607.75	638.14

C. Andrew drops a rubber ball from a height of 2 m. After each bounce, the height of the ball's bounce is one-half the height of the ball's previous bounce.

Bounce	1	2	3	4	5
Height of Ball After Each Bounce (m)	1	0.5	0.25	0.125	0.063

D. David tosses a coin repeatedly. Each time he tosses the coin, he records 1 if the coin comes up heads and 0 if the coin comes up tails.

Students will have different results.

E. A square number can be represented by an arrangement of dots. Each square number is the total number of dots in each diagram. Here are the first four square numbers.

Diagram	1	2	3	4	5
Square Number	1	4	9	16	25

- F.** In a fractal, a pattern repeats itself in the same shape. The Sierpinski triangle is an example of a **fractal**. Divide one equilateral triangle into four, congruent triangles by joining the midpoints of the sides of the larger triangle and remove the centre triangle. You can repeat these steps many times. Here are the first four stages of the Sierpinski triangle.

There are several ways of counting the triangles. The total number of triangles, for example in stage 2, includes the three solid triangles, the “white” triangle, and the largest triangle. Alternatively, students may count only the solid triangles.

Stage	1	2	3	4	5
Total Number of Triangles	1	5	17	53	161
Total Number of Solid Triangles	1	3	9	27	81

- 2.** Which of the relationships are predictable? unpredictable?

All of the relationships are predictable, except D.

- 3.** If possible, describe in words the pattern in each relationship.

Students can describe the patterns in various ways. Encourage them to check that their description predicts all values and, where possible, to simplify descriptions.

- A.** Multiply the number of vacuum cleaners sold by 110 and add 400.
- B.** Multiply 500 by 1.05, which is multiplied by itself n times, where n is the number of years.
- C.** Multiply 2 by 0.5, which is multiplied by itself n times, where n is the number of bounces.
- E.** Multiply the diagram number by itself.
- F.** Total number of triangles: Three times the previous number plus 2, which describes a recursive relationship. Here is an alternative description: multiply 2 by 3 raised to the power of n less 1 and then subtract 1 from the product. n is the stage number.

Total number of solid triangles: Each stage is a power of 3 and the exponent is 1 less than the stage number.

- 4.** If possible, use the pattern to find the next three numbers in each list.

A.

Number of Vacuum Cleaners Sold	6	7	8
Earnings (\$)	1060	1170	1280

B.

At End of Each Year	6	7	8
Money in Savings Account (\$)	670.05	703.55	738.73

C.

Bounce	6	7	8
Height of Ball After Each Bounce (m)	0.03	0.02	0.01

D. not possible

E.

Diagram	6	7	8
Square Number	36	49	64

F.

Stage	6	7	8
Total Number of Triangles	485	1457	4373
Total Number of Solid Triangles	243	729	2187

- 5.** Do any of these relationships come to an end? Explain.

Theoretically, the pattern in C would continue forever; in reality, a ball would eventually come to rest (bounce height = 0). The number of vacuums in A could be restricted by supply, but theoretically the pattern could continue forever. The other patterns could also continue indefinitely theoretically. The solid triangles in F would not be visible for large values of the independent variable.

6. For each relationship and if possible, find an algebraic expression that describes the value of each term in the list according to its position in the list. Let n represent the position of the term in the list, where n is a natural number, $\{1, 2, 3, \dots\}$.

A. $t_n = 400 + 110n$

B. $t_n = 500(1.05)^n$

C. $t_n = 2(0.5)^n$

D. no pattern

E. $t_n = n^2$

F. pattern 1: $t_n = 3(t_{n-1}) + 2$

or $t_n = 2(3n - 1) - 1$

pattern 2: $t_n = 3^{n-1}$

7. Use the algebraic expression you found in step 6 to determine the tenth number in each list.

A. $t_{10} = 400 + 110(10) = 1500$

B. $t_{10} = 500(1.05)^{10} = 814.45$

C. $t_{10} = 2(0.5)^{10} = 0.002$

E. $t_{10} = 10^2 = 100$

F. pattern 1: $3(t_{n-1}) + 2$

$t_{10} = 3t_9 + 2$

$t_{10} = 3(3t_8 + 2) + 2$

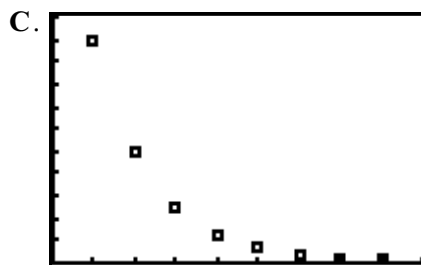
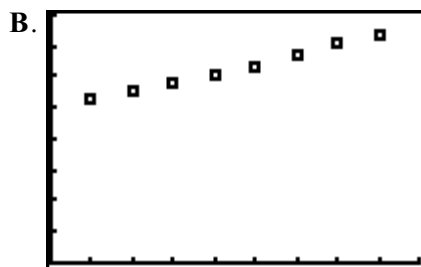
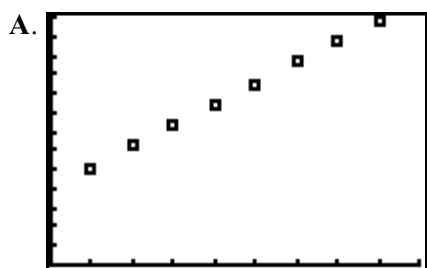
$t_{10} = 3[3(4373) + 2] + 2 = 39\,365$

or

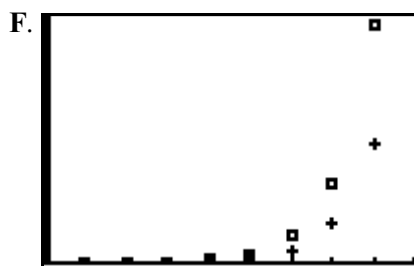
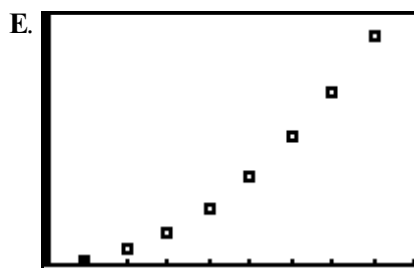
$t_{10} = 2(39) - 1 = 39\,365$

pattern 2: $t_{10} = 39 = 19\,683$

8. If n is the independent variable, then graph each predictable relationship. Can you join the points with a line or a curve? Explain.



D. not predictable



For part F, one graph is a transformation of the other.

In each case, the points in the graph cannot be joined by a line or a curve because the domain is the set of natural numbers. The data is then discrete. A curve of best fit could be used to show a trend but the curve does not indicate that the set of points is continuous or dense.

ASSESSMENT SUGGESTION

With or without using technology, students might provide a table, a graph, and an algebraic expression to show the predictability of the pattern in a sequence.

EXTENSION SUGGESTION

Ask students to create a sequence and the corresponding algebraic expression. Have them provide a partner with four consecutive terms of this sequence and challenge the partner to describe the relation and predict other terms.

Focus 1.1

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Example 1

Students may need to develop some typical patterns, such as common differences, common factors, and so on, to help them identify patterns in a sequence. However, they must accept that the pattern in a list of numbers is not necessarily predictable.

Example 2

Any specific term can be generated by substituting the number of the term into the expression for the general term. Ensure students use good form in their responses—showing the general term, using the subscript to denote a specific term, showing the substitution, and aligning equal signs.

Example 3

Deciding whether a sequence is finite or infinite is easy for most students when they think about the sequence continuing.

Example 4

Some students may need to first calculate the area of the second triangle as $\frac{1}{2}bh$ and then

see that this area is $\frac{1}{4}$ the area of the original triangle. This example connects all the ideas of this section—recognizing a pattern, determining a general term, generating specific terms, and graphing the relationship between the term's number and its value. It may be

helpful to describe the points generated by the relationship as (n, t_n) .

Additional Example: Functions

As the players on the team are introduced over the public address system at a baseball game, each player runs onto the field and slaps hands with each teammate. The second player introduced slaps one pair of hands, the third player introduced slaps two pairs of hands, and so on.

- (a) How many pairs of hands does the tenth player slap?
- (b) Create an expression to describe the number of hand slaps that any player would make.

Solution

- (a) The tenth player will slap nine pairs of hands.
- (b) $t_n = n - 1$

Additional Example: Functions and Relations

Pete has saved \$4500 from his summer job to pay for living expenses at university. He anticipates that his expenses, including rent and groceries, will average \$200 per week.

- (a) Create Pete's savings account if he opens it at the beginning of September. List the balance after each transaction.
- (b) Describe each term of the list in (a) according to the week it represents.
- (c) What problem will Pete face?

Solution

- (a) \$4500, \$4300, \$4100, \$3900, ...
- (b) $t_n = 4500 - 200(n - 1)$
- (c) Pete's savings will be used up before the year ends.

Practise, Apply, Solve 1.1

Questions with a Functions and Relations Focus:

1–11, 13, 14, 18, 19, 23

Questions with a Functions Focus:

1–8, 10, 14–19

6. Knowledge and Understanding, p. 22

Management Suggestion

Ask students to hand in their solutions to this question and, while they continue to work on other questions, quickly scan the work and provide verbal comments to individuals.

Achievement Chart Connection

Students must demonstrate they understand the role of a general term in generating the terms of a sequence.

Level 3 Performance

$$t_n = (5 - n)^n$$

$$t_1 = (5 - 1)^1 = 4$$

$$t_2 = (5 - 2)^2 = 9$$

$$t_3 = (5 - 3)^3 = 8$$

$$t_4 = (5 - 4)^4 = 1$$

$$t_5 = (5 - 5)^5 = 0$$

$$t_{10} = (5 - 10)^{10} = 9\,765\,625$$

ASSESSMENT SUGGESTION

Knowledge and Understanding Quick Quiz

Ask students to generate specific terms from a given general term and to create a general term for a sequence.

ASSESSING LEARNING SKILLS

Teamwork and Independence

Observe students as they work among themselves. Do they cooperate or do they work individually? Are they able to distinguish when it is appropriate to work together or individually?

10. Communication, p. 22

Management Suggestion

Discuss the question as a class after students consider their individual responses.

Achievement Chart Connection

Students may use mathematical language to explain their reasoning.

Level 3 Performance

The terms of a sequence are distinct numbers, and no other numbers are possible between two consecutive terms in the ordered list. Thus, the graph of a sequence is a collection of distinct points, one point for each number in the sequence.

14. Application: p. 23

Management Suggestion

Ask groups of three or four students to compare their work and present a “best solution” using the overhead. Select one or two solutions and discuss each solution with the class.

Achievement Chart Connection

Students use a sequence to describe the familiar idea of bank balances.

Level 3 Performance

(a)

Year	Amount
1	\$1045.00
2	\$1092.03
3	\$1141.17

(b) $t_n = 1000(1.045)^n$

(c) $t_{10} = 1000(1.045)^{10} = \$1552.97$

18. Thinking, Inquiry, Problem Solving, p. 24

Management Suggestion

Allow students to work with a partner or in a small group to discuss features of the pattern and develop a general term. Ask them to note all of the general terms they try, even the ones they reject.

Achievement Chart Connection

Students must find a pattern that is not immediately obvious. They may create an algebraic model of the general term and then adjust it until it accurately describes the terms.

Level 3 Performance

Some students may find it helpful to create a table and search for the pattern and general term before predicting the seventh term.

(a) $t_1 = 2, t_2 = 8, t_3 = 18, t_4 = 32, t_7 = 98$

(b) $t_n = 2n^2$

19. Check Your Understanding: p. 24

Management Suggestion

Ask students to create an acronym or nonsense verse to help them remember the steps for finding sequences and general terms.

EXTENSION SUGGESTION

Knowledge and Understanding Quick Quiz

Invite students to describe scenarios where a few numbers can be counted or calculated. Then ask them to find a general term and make useful predictions. Medicine or wildlife management applications might be suggested.

The Chapter Problem—Controlling Non-Native Plant Populations

By the end of this section, students should be able to develop several elements in a sequence and look for a pattern so that they may create a general term for the sequence. Some students might consider creating a spreadsheet to generate data.

Students may be surprised by the increasingly rapid growth in the numbers. They should recognize that the growth is not linear and may be able to recognize that each term is an approximate multiple of the previous term.

The Chapter Problem Answers can be found on pp. 1-54 to 1-56 of the *Teacher Resource*.

Management Suggestion

Select just one challenge, depending upon student interests and availability of reference material. Challenge 2 may offer more opportunity for tactile or visual learning, and may be more appealing to Functions students.

Challenge 1

Remind students that they should be able to

- calculate several elements in a sequence
- develop a spreadsheet to generate data for investigation
- look for a pattern in the data to create a general term

Challenge 2

At this point, students could draw sample diagrams and generate a few terms of a sequence to have data for finding a pattern.