

2.3 Geometric Series

Specific Expectations

The following expectation is common to both the Functions and Relations and Functions courses.

- FA1.05 determine the sum of the terms of an arithmetic or a geometric series, using appropriate formulas and techniques

Evidence of Learning

Students should be able to:

- generate a geometric series
- calculate the sum of a geometric series
- select the appropriate sum formula

Text Page 119

Time Required Functions and Relations or Functions: 75 min

Materials and Equipment

- scientific calculator

Overview

- Define a geometric series.
- Use Gauss's method and a common ratio to calculate the sum of a geometric series.
- Develop the formula for the sum of a geometric series.

Management Options

Groups – Pairs or teams can work through all the steps of the *Think, Do, Discuss* with extensive teacher support for steps 6 to 9.

Teacher Demonstration – Lead Functions and Relations students through developing the formula for the sum of a geometric series. Show other students that Example 1 (a) works when the terms of the sequence are added or when either of the sum formulas is applied.

Independent Work – Allow individual students to practise “reading” a math text in a supportive environment. Circulate and ask questions to check understanding.

MANAGEMENT SUGGESTION

Suggest that it is good practice to state the series and list the variables for every problem.

Geometric Series

Think, Do, Discuss, p. 119

Students will solve problems involving the sum of a geometric sequence, that is, a series. Students in the Functions and Relations course should be able to algebraically develop the formula for the sum of a geometric series by following the method suggested in question 15 in the *Practise, Apply, Solve*. Drawing tree diagrams for step 1 of the *Think, Do, Discuss* and for question 9 of the *Practise, Apply, Solve* is good practice for counting problems in MGA4U (grade 12). Example 4 provides an algebraic solution to a problem, which is similar to the experiment in section 2.2.

1. *Start with the company's president at the top, and draw a diagram of the phone tree for the first four rounds of calls. The diagram represents the sequence of the number of employees notified at each round.*
2. *How many employees were notified by the president, who made the first round of calls? How many employees were notified during the second round of calls? the third round of calls? the fourth round of calls?*

Three employees were notified on the first round of calls, 9 during the second, 27 on the third, and 81 on the fourth.

3. The number of employees notified during each round of calls forms a sequence. What do you call this sequence? Explain. Determine the general term, t_n , to represent the number of employees notified during the n th round.

The sequence is 3, 9, 27, 81, ... This sequence is geometric, because each term is generated by multiplying the previous term by the common ratio 3. For $a = 3$ and $r = 3$,

$$\begin{aligned}t_n &= ar^{n-1} \\t_n &= 3(3)^{n-1} \\t_n &= 3^n\end{aligned}$$

4. Write the sequence that represents the number of employees notified for the first seven rounds of calls. Find the total number of employees notified after the first seven rounds of calls.

The sequence 3, 9, 27, 81, 243, 729, 2187 represents the employees notified in the first seven rounds. A total of 3279 employees were notified in seven rounds.

5. The sum of the terms of a geometric sequence is called a **geometric series**. The sum of the sequence in step 4 is S_7 , where $S_7 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7$. Write the series, substituting the appropriate values for t_1 to t_7 .

$$S_7 = 3 + 9 + 27 + 81 + 243 + 729 + 2187$$

6. To develop a formula for the sum of the geometric series, begin by multiplying both sides of the equation in step 5 by the common ratio, $r = 3$. Write the terms in $3S_7$ so that t_1 of $3S_7$ is below t_2 of S_7 and t_2 of $3S_7$ is below t_3 of S_7 , and so on. Compare S_7 to $3S_7$. What is the same? What is different? Would there be so many common terms if you had multiplied S_7 by a number other than the common ratio of $r = 3$? Explain.

$$S_7 = 3 + 9 + 27 + 81 + 243 + 729 + 2187$$

$$3S_7 = 3(3) + 3(9) + 3(27) + 3(81) + 3(243) + 3(729) + 3(2187)$$

The second term of S_7 is the same as the first term of $3S_7$. $3S_7$ is like S_7 , but $3S_7$ starts with the second term of S_7 and ends with what would be the next term of S_7 . If you had multiplied by a power of 3, that is, 9 or 27, some terms would be the same, but not as many. If you had multiplied by any

other numbers, none of the terms would be the same.

7. Subtract S_7 from $3S_7$. What values remain on the right side? Which terms of the geometric sequence do these values represent?

$$S_7 = 3 + 9 + 27 + \dots + 2187$$

$$3S_7 = 3(3) + 3(9) + \dots + 3(729) + 3(2187)$$

$$2S_7 = -3 \qquad \qquad \qquad -3(2187)$$

-3 is $-t_1$ and $3(2187)$ is $3(t_7)$ or $r(t_n)$.

8. $2S_7$ is now the sum of only two terms. What must you do to both sides so that the left side is S_7 ? Find S_7 . What does this sum represent?

Dividing both sides by 2 yields

$$S_7 = \frac{-3 + 3(2187)}{2}, \text{ which represents the}$$

sum of the first seven terms. This sum should equal the sum in step 5.

9. Use the method in steps 7 and 8 to determine the total number of employees notified after ten rounds of calls.

The first ten terms are 3, 9, 27, ..., t_{10} , and $a = 3$, $r = 3$, $n = 10$, $t_{10} = ?$, and $t_n = 3^n$ (from step 3).

$$t_n = 3^{10}$$

$$\begin{array}{r} 3S_{10} = 3(3) + 3(9) + \dots + 3(3)^9 + 3(3)^{10} \\ S_{10} = 3 \qquad + 9 \qquad + 27 + \dots \qquad + 3^{10} \end{array}$$

$$2S_{10} = -3 \qquad \qquad \qquad + 3(3)^{10}$$

$$S_{10} = \frac{-3 + 177147}{2}$$

$$= 88572$$

88 572 employees were called after ten rounds.

10. The general term of a geometric sequence is $t_n = 3(4)^{n-1}$. Use the method in steps 5 to 8 to find S_8 , the geometric series, or sum, of the first eight terms.

$$t_n = 3(4)^{n-1}$$

$$\begin{aligned}t_1 &= 3(4)^{1-1} \\ &= 3\end{aligned}$$

$$\begin{aligned}t_2 &= 3(4)^{2-1} \\ &= 12\end{aligned}$$

$$\begin{aligned}t_3 &= 3(4)^{3-1} \\ &= 48\end{aligned}$$

$$a = 3, r = 4, \text{ and } t_8 = 3(4)^{8-1} \text{ or } 49152$$

$$4S_8 = 4(3) + 4(12) + \dots + 4(12\,288) + 4(49\,152)$$

$$S_8 = 3 + 12 + 48 + \dots + 49\,152$$

$$3S_8 = -3 + 4(49\,152)$$

$$S_8 = \frac{196\,605}{3}$$

$$= 65\,535$$

11. A geometric sequence has the general term $t_n = ar^{n-1}$, and a and r are known. Suggest a formula for finding S_n , the geometric series of this sequence.

$$S_n = \frac{-t_1 + rt_n}{(r-1)}$$

MANAGEMENT SUGGESTION

See question 15 of the *Practise, Apply, Solve* on text p. 125 for a formal development of the formula.

12. List the first five terms of the series if the first term is a and the common ratio, r , is 1. What is the sum of these five terms, S_5 ?

The first five terms of this series are
 $a + a(1) + a(1)^2 + a(1)^3 + a(1)^4$.

$$a + a(1) + a(1)^2 + a(1)^3 + a(1)^4$$

$$= a + a + a + a + a$$

$$= 5a$$

MANAGEMENT SUGGESTION

Key Ideas

Draw students' attention to the difference between the formula for the n th term of a geometric sequence, ar^{n-1} , and the sum of a geometric series, $\frac{a(r^n - 1)}{r - 1}$. In the first formula, the exponent is $n - 1$, but in the second formula, the exponent is just n .

Additional Example: Functions and Relations

A series can be written in abbreviated form, called **sigma notation**, using the Greek letter Σ (sigma), which is called the summation sign. For example, the series $3 + 9 + 27 + \dots +$

19 683 can be written as $\sum_{n=1}^9 3^n$. In this case,

$\sum_{n=1}^9 3^n = 3^1 + 3^2 + 3^3 + \dots + 3^9$. The general term, 3^n , is the **summand**, and the letter n is the **index**. Any letter can represent the index.

1. Express each geometric sequence in sigma notation.

(a) $2 + 4 + 8 + 16 + \dots + 4096$

(b) $6 + 12 + 24 + 48 + \dots + 98\,304$

(c) $2 + 10 + 50 + 250 + \dots + 156\,250$

Solution

(a) $a = 2$, $r = 2$, and $t_n = 4096$

$$t_n = ar^{n-1}$$

$$t_n = 2(2)^{n-1}$$

$$t_n = 2^n$$

$$4096 = 2^n \text{ and } 2^{12} = 2^n.$$

Therefore, $n = 12$.

$$\sum_{n=1}^{12} 2^n = 2 + 4 + 8 + 16 + \dots + 4096$$

(b) $a = 6$, $r = 2$, and $t_n = 98\,304$

$$t_n = ar^{n-1}$$

$$t_n = 6(2)^{n-1}$$

$$t_n = 3(2)(2)^{n-1}$$

$$t_n = 3(2)^n$$

$$t_n = 3(2)^n$$

$$98\,304 = 3(2)^n$$

$$32\,768 = 2^n$$

$$2^{15} = 2^n$$

Therefore, $n = 15$.

$$\sum_{n=1}^{15} 3(2)^n = 6 + 12 + 24 + 48 + \dots + 98\,304$$

(c) $a = 2$, $r = 5$, and $t_n = 156\,250$

$$t_n = ar^{n-1}$$

$$t_n = 2(5)^{n-1}$$

$$156\,250 = 2(5)^{n-1}$$

$$78\,125 = 5^{n-1}$$

$$5^7 = 5^{n-1}$$

Therefore, $7 = n - 1$ and $n = 8$.

$$\sum_{n=1}^8 2(5)^{n-1} = 2 + 10 + 50 + 250 + \dots + 156\,250$$

2. How would you express an arithmetic series in sigma notation? Explain. Support your explanation with an example.

Solution

An arithmetic series can be expressed in sigma notation, since each subsequent term is found by adding the common difference to the previous term.

For the sequence $5 + 8 + 11 + \dots + 35$, $a = 5$, $d = 3$, and $t_n = 35$.

$$\begin{aligned} t_n &= 5 + (n - 1)(3) \\ t_n &= 2 + 3n \quad \text{Substitute.} \\ 35 &= 2 + 3n \\ 11 &= n \\ \sum_{n=1}^{11} 2 + 3n &= 5 + 8 + 11 + \dots + 35 \end{aligned}$$

3. Determine S_{10} , where $\sum_{n=1}^{10} \left(\frac{-1}{2}\right)^n$.

Solution

$$\sum_{n=1}^{10} \left(\frac{-1}{2}\right)^n = \left(\frac{-1}{2}\right)^1 + \left(\frac{-1}{2}\right)^2 + \dots + \left(\frac{-1}{2}\right)^{10},$$

$$a = \left(\frac{-1}{2}\right)^1, r = \left(\frac{-1}{2}\right)^1, \text{ and } n = 10$$

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_{10} &= \frac{\frac{-1}{2} \left[\left(\frac{-1}{2}\right)^{10} - 1 \right]}{\left(\frac{-1}{2}\right) - 1} \\ &= \frac{\frac{-1}{2} \left[\left(\frac{1}{1024}\right) - \frac{1024}{1024} \right]}{\left(\frac{-3}{2}\right)} \\ &= \left(\frac{-1}{2}\right) \left(\frac{2}{-3}\right) \left(\frac{-1023}{1024}\right) \\ &= \frac{-341}{1024} \end{aligned}$$

Additional Example 1: Functions

To encourage Dan to do his chores, his parents offer to double his allowance every week, but he earns only one cent the first week.

- (a) When will Dan earn enough in one week to buy a movie-theatre ticket (about \$10)?

- (b) When will Dan earn \$100 in total?

- (c) Is the offer reasonable?

Solution

- (a) The sequence 1, 2, 4, 8, 16, ... represents Dan's pay in cents for weeks 1, 2, 3, 4, 5, respectively.

$$a = 1, r = 2, t_n = \$10 \text{ or } 1000\text{¢}, \text{ and } n = ?$$

$$t_n = ar^{n-1}$$

$$1000 = 1(2)^{n-1}$$

But $2^9 = 512$ and $2^{10} = 1024$. Then 2^{10} is about equal to 2^{n-1} . Therefore, $10 = n - 1$ and $n = 11$. Dan will earn \$10.24 in week 11.

- (b) $a = 1, r = 2, S_n = \$100$ or 10 000¢, and $n = ?$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$10\,000 = \frac{1(2^n - 1)}{2 - 1}$$

$$10\,000 = 2^n - 1$$

$$10\,001 = 2^n$$

$$\text{But } 2^{13} = 8192 \text{ and } 2^{14} = 16\,384.$$

Dan will earn \$81.92 in 13 weeks and \$163.84 in 14 weeks.

- (c) At the beginning, the pay is not reasonable, but in week 20, Dan could earn \$5242.88, providing his parents are willing to pay this amount.

Additional Example 2: Functions

The fifth term of a geometric series is 48 and the eighth term is 384. Calculate the sum of the first ten terms of this series.

Solution

$$t_5 = 48, t_8 = 384, a = ?, r = ?, \text{ and } S_{10} = ?$$

To calculate two variables, you need two equations. Substitute in $t_n = ar^{n-1}$.

$$\text{For } t_8, ar^7 = 384 \text{ ①}$$

$$\text{For } t_5, ar^4 = 48 \text{ ②}$$

Divide equation ① by equation ②. (See Note that follows.)

$$r^3 = 8$$

$$r = 2$$

Substitute $r = 2$ in equation ②.

$$a(2)^4 = 48$$

$$a = 3$$

The series is $3 + 6 + 12 + \dots$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\begin{aligned} S_{10} &= \frac{3(2^{10} - 1)}{2 - 1} \\ &= 3(1024 - 1) \\ &= 3069 \end{aligned}$$

Therefore, the sum of the series is 3069.

Note: Compare this solution to the solution in Example 4, section 2.1 (text p. 113). In an arithmetic sequence or series, *adding* the common difference, d , creates the successive terms: to solve for d , *subtract* one equation from another. In a geometric sequence or series, *multiplying* by the common ratio, r , creates the successive terms: to solve for r , *divide* one equation by another.

Practise, Apply, Solve 2.3

Questions with a Functions and Relations

Focus:

1–5, 8, 9, 11–13, 15–17, 19–21

Questions with a Functions Focus:

1–8, 10, 12, 14, 15, 17, 18

4. Knowledge and Understanding, p. 123

Management Suggestion

Ask students to work in pairs to establish the source of the variables before substituting. Pairs can verify their results with the other pairs.

Achievement Chart Connection

Students identify the variables, select the proper formula, and should compute correctly.

Level 3 Performance

Students should select the appropriate formula and state it correctly, assign variables correctly and substitute them in the formula, and calculate correctly.

8. Communication, p. 124

Management Suggestion

Suggest students imagine describing the solution to a student in grade 7 and then to a classmate who is absent. Teams can brainstorm for five minutes before writing the solution. In the team, students can edit other students' solutions. Each team can select one explanation to be read to the class.

Achievement Chart Connection

Students should correctly apply the terms to the problem and clearly explain the methods.

Level 3 Performance

Students should generate the series to match the context of the question and identify the variables. They should demonstrate adequate understanding of the concepts of “term” and “sum” and clearly communicate the steps for both methods.

12. Application, p. 125

Management Suggestion

Students can work to create the series and identify the variables before substituting values in formulas. They can then verify their results with other partners.

Achievement Chart Connection

Students create a series and identify the variables from the context of the question. Then they select a formula for each part of the question and calculate.

Level 3 Performance

Students should create the series, select the appropriate formulas, and state them correctly. They need to assign variables correctly, substitute values in the formulas, and calculate correctly.

15. Thinking, Problem Solving, Inquiry, p. 125

Management Suggestion

Direct Functions and Relations students in developing this formula.

When developing the geometric series sum formula, multiply S_n by r . When developing the arithmetic series sum formula, add S_n twice. Discuss.

Students should first solve question 17(a).

Achievement Chart Connection

Students can duplicate the method in the *Think, Do, Discuss* using the given terms. The reasoning should be logical and consistent with the context.

Level 3 Performance

Students' solutions should be clearly organized and demonstrate logical reasoning that supports the conclusion. There should be evidence that students recognize the patterns.

17. Check Your Understanding**Management Suggestion**

Guide the first student or pair that seeks direction. Then these student(s) can help the next student or pair who have questions, and so on. Students clarify and consolidate their learning when they teach, and explain to, their peers. Students will often be satisfied to see that adding the terms produces the same answer as applying the formula. Functions and Relations students must experience formal proofs; see also question 15.

Achievement Chart Connection

Students can duplicate the method in the *Think, Do, Discuss* using the given terms. The reasoning should be logical and consistent with the context.

The Chapter Problem—Financial Planning

The Chapter Problem Answers are on p. 2-67 of the *Teacher Resource*.

Challenge 1

Students could respond to this challenge at any time through research, but the answers will make more sense if they do this challenge when they do section 2.12.

When students research the Chapter Problem, remind them to read Challenge 1 and to research mortgages as well.

Challenge 2

The challenge will make sense after they have become familiar with the financial terminology, although section 1.8 provides the tools for calculating. Assign this challenge after section 2.8.

The Challenge Answers are on p. 2-82 of the *Teacher Resource*.