

2.6 *Technology:* TI-83 Plus Calculator: Finding the Sum and the Cumulative Sum of a Series

Text Page 143

Time Required Functions and Relations or Functions: 30 min

Materials and Equipment

- TI -83 Plus calculator

Overview

- Calculate the sum of a series, using a function of the TI-83 Plus.
- Calculate the cumulative sum of a series, using a function of the TI-83 Plus.

Finding the Sum and the Cumulative Sum of a Series

Suggest to students that they may use the **sum** and **sequence** functions for checking answers.

Management Options

Teacher Demonstration – Lead students through the lesson using the TI-83 Plus and the view screen.

Independent Work – If students have access to a graphing calculator, you could assign this section as homework.

ASSESSING LEARNING SKILLS

If students work independently on the lesson, are students able to decide when they need extra help or do they wait until you ask them if they need help?

Questions with a Functions and Relations Focus:

1, 2

Questions with a Functions Focus:

1, 2

2.7 Using Series to Analyze Financial Situations: Future Value

Specific Expectations

The following expectations are common to the Functions and Relations course and the Functions course.

- FA2.01 derive the formulas for compound interest and present value, the amount of an ordinary annuity, and the present value of an ordinary annuity, using the formulas for the n th term of a geometric sequence and the sum of the first n terms of a geometric series;
- FA2.03 solve problems involving the amount and the present value of an ordinary annuity
- FA3.01 analyse the effects of changing the conditions in long-term savings plans (e.g., altering the frequency of deposits, the amount of deposit, the interest rate, the compounding period, or a combination of these) (Sample problem: Compare the results of making an annual deposit of \$1000 to an RRSP, beginning at age 20, with the results of making an annual deposit of \$3000, beginning at age 50)
- OC3.04 demonstrate the correct use of mathematical language, symbols, visuals (e.g., diagrams, graphs), and conventions

Evidence of Learning

Students should be able to

- design an informative time line for the future value of an annuity
- understand the connection between a geometric series and the future value of an annuity
- apply the future value formula for an ordinary simple annuity

Text Page 145

Time Required Functions and Relations:
75 min

Time Required Functions: 110 min

Materials and Equipment

- TI-83 Plus calculator

Overview

- Apply the sum of a geometric series to financial situations.
- Develop a formula for the future value or amount of an ordinary simple annuity.
- Represent the accumulating amount with a time line.

Future Value

Students in the Functions and Relations course should be able to algebraically develop the formula for the future value of an ordinary simple annuity. They should recognize the connection between this formula and the formula for the sum of a geometric series (see *Key Ideas*, text p. 147). You might discuss question 14 in the *Practise, Apply, Solve*. Students who develop their skills in algebra are better able to check the reasonableness of answers produced using technology.

MANAGEMENT SUGGESTION

When students solve for the future value, or the present value, of an annuity, it is good practice to draw a time line, write the series, and list the values of the variables for the appropriate formula.

Management Options

Groups – Pairs may work through the *Think, Do, Discuss* of each part.

Teacher-Led Demonstration – Less confident students might prefer a teacher-led demonstration.

Independent Work – For Part 1, students could practise “reading” a math text in a supportive environment. Answer the question of the first student who asks and then ask that student to help another classmate who has the same question. Reconvene as a class before proceeding to Part 2. Students may need assistance with step 1 of the *Think, Do, Discuss* in Part 2.

TECHNOLOGY TIP

It is critical that students use a scientific calculator’s memory efficiently. Remind students to round only the final answer. Banks use a *rounding value* of 6 (see section 2.11).

Part 1: Representing an Ordinary Simple Annuity with a Series

Think, Do, Discuss, p. 145

- Each of the symbols from $F1$ to $F7$ represents the amount, or future value, of the payment at the end of seven years. What is the amount of the last deposit, $F1$?

The amount of the last deposit, $F1$, is \$3000. Because this deposit was made at the end of the period, it does not earn interest.

- At the end of the sixth year, Mr. Watts deposits \$3000. For how many periods is this deposit in the account? Write an expression to represent $F2$, the future value of the deposit made at the end of the sixth year.

The deposit made at the end of the sixth year has been in the account for one interest period.

$$A = P(1 + i)^n, \text{ and } A = F2, P = 3000, \\ i = 7.5\% \text{ or } 0.075, \text{ and } n = 1$$

$$F2 = 3000(1 + 0.075)^1 \\ = 3000(1.075)^1$$

- Repeat step 2 for each of the other deposits, $F3$ to $F7$.

$$A = P(1 + i)^n, \text{ and } A = Fn, P = 3000, \\ i = 7.5\% \text{ or } 0.075, \text{ and } n = 2, 3, 4, 5, 6 \text{ for } F3 \text{ to } F7$$

$$F3: A = 3000(1.075)^2$$

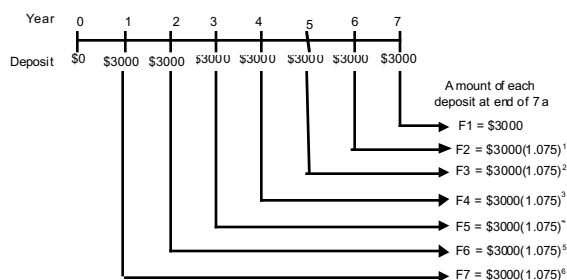
$$F4: A = 3000(1.075)^3$$

$$F5: A = 3000(1.075)^4$$

$$F6: A = 3000(1.075)^5$$

$$F7: A = 3000(1.075)^6$$

- Copy and complete the time line.



- Write the series that represents the future values or amounts of all the deposits at the end of seven years. Use the expressions that you found in steps 1 to 3. Begin with $F1$.

$$3000 + 3000(1.075)^1 + 3000(1.075)^2 \\ + 3000(1.075)^3 + 3000(1.075)^4 \\ + 3000(1.075)^5 + 3000(1.075)^6$$

- Describe this type of series. What is the first term? Describe the first term in the context of the investment. What is the common ratio? Describe the common ratio in the context of the investment. How many terms are in the series?

The series is geometric, and $t_1 = 3000$, which is the last deposit. The common ratio, r , is $(1.075)^1$, which is the factor by which each successive deposit increases and which represents interest. There are seven terms in the series.

MANAGEMENT SUGGESTION

Students need to carefully choose the values of the variables when using $A = P(1 + i)^n$ and

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ in the same question.}$$

7. Recall the formula for the sum of this type of series. Find the sum of the future values of the deposits. Compare your solution to the one you found in the Think, Do, Discuss of Part 1 in section 2.5.

$$S_n = \frac{a(r^n - 1)}{r - 1}, \text{ and } a = 3000, r = (1.075)^1$$

or 1.075, and $n = 7$ (the number of deposits)

$$\begin{aligned} S_7 &= \frac{3000[(1.075)^7 - 1]}{1.075 - 1} \\ &= \frac{3000(1.659\,049\,14 - 1)}{0.075} \\ &= 26\,361.97 \end{aligned}$$

\$26 361.97 is the sum of the deposits, which is the same amount in cell E8 of the spreadsheet on page 129.

Part 2: Developing a Formula for the Amount of an Ordinary Simple Annuity

Think, Do, Discuss, p. 146

1. What does i represent in the time line? Explain why the exponent in the future value of the first payment is $n - 1$.

The variable i represents the interest rate per interest period. The first deposit made at the *end* of the first year earns interest not for the first period but for each successive period, because the interest is also calculated at the *end* of each period. So, for the example in the first part, the first deposit earns interest for six periods, not seven periods, and the second deposit earns for five periods, not six periods, and so on. Therefore, the exponent is $n - 1$, not n .

2. Given Andrea's time line, write the series of terms, S_n , that represents the future value or amount of each payment of the annuity, beginning with the n th payment. What type of series is S_n ?

The series is

$$\begin{aligned} S_n &= R + R(1 + i) + R(1 + i)^2 + \dots \\ &+ R(1 + i)^{n-2} + R(1 + i)^{n-1}, \text{ which is} \\ &\text{geometric.} \end{aligned}$$

3. What is the first term of the series? What is the common ratio? Substitute these values in the formula you used in step 7 of Part 1. Simplify the denominator.

$t_1 = R$ and $r = (1 + i)$. Also, $a = R$ and the variable n is the same.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{R[(1 + i)^n - 1]}{(1 + i) - 1}$$

$$S_n = \frac{R[(1 + i)^n - 1]}{i}$$

$$\text{or } S_n = R \times \frac{(1 + i)^n - 1}{i}$$

4. Use this formula again to find the amount of an annuity if the amount of each payment is \$1000, payable at the end of every six months for three years. The interest rate is 10%/a, compounded semiannually.

Substitute $R = 1000$, $i = 10\% \div 2$ or 5%, and $n = 6$ (Interest is compounded twice a year for 3 years.)

$$S_n = R \times \frac{(1 + i)^n - 1}{i}$$

$$\begin{aligned} S_6 &= 1000 \times \frac{(1 + 0.05)^6 - 1}{0.05} \\ &= 6801.91 \end{aligned}$$

The amount of this annuity is \$6801.91.

MANAGEMENT SUGGESTION

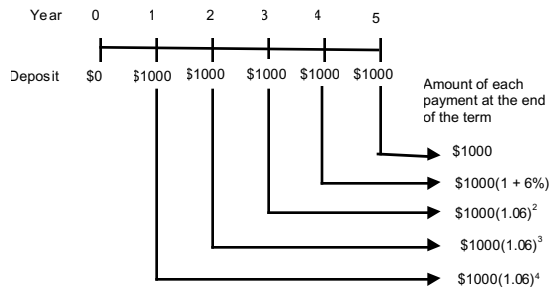
Most students will memorize formulas through frequently using them, but encourage students to record the formulas in their notebooks or study journals and to state or write the formula each time they need it so that they will learn how and when to use formulas. Test students' understanding by providing the formula and asking them to define the variables.

Additional Example: Functions and Relations

Chris deposits \$1000 at the end of every year for five years in an account that pays 6%/a, compounded annually. His sister Kerrie does the same, except the interest for her account is compounded semiannually. Whose account earns more interest and by how much? Explain.

Solution

Chris's Account. Draw a time line.



$a = 1000$, $r = (1.06)^1$ or 1.06, and $n = 5$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

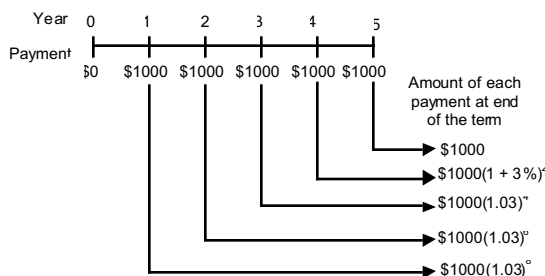
$$S_5 = \frac{1000[(1.06)^5 - 1]}{1.06 - 1}$$

$$= \frac{1000(0.33822)}{0.06}$$

$$= 5637.09$$

The accumulated amount in Chris's account is \$5637.09.

Kerrie's Account: Draw a time line. The interest is compounded semiannually.



$$S_n = \frac{a(r^n - 1)}{r - 1}, \text{ and } a = 1000,$$

$$r = (1.03)^2, \text{ and } n = 5 \text{ (the number of deposits)}$$

$$S_5 = \frac{1000\{[(1.03)^2]^5 - 1\}}{(1.03)^2 - 1}$$

$$= \frac{1000[(1.03)^{10} - 1]}{1.0609 - 1}$$

$$= \frac{1000(0.3439163793)}{0.0609}$$

$$= 5647.23$$

The accumulated amount in Kerrie's account is \$5647.23. Kerrie's account earns \$10.14 more in interest, because the interest is compounded more frequently. A small amount of interest is added to the principal halfway through each year, so this account has a greater principal at the end of each year and the interest calculated on this principal is also greater.

MANAGEMENT SUGGESTION

This approach is far more complicated when there are fewer interest intervals than payment intervals. Section 2.10 shows a practical method for either case.

Additional Example: Functions

Both Mike Miser and his best friend, Perry Procrastinator, just turned 20. Mike decides to plan for his future by depositing \$3000 at the end of each year in an investment paying 9%/a, compounded annually, until his 40th birthday. After this birthday, he stops making deposits and leaves the balance in the account to collect interest until his 55th birthday. Perry waits until his 30th birthday to start planning for his future. To make up for lost time, he decides to deposit \$6000 at the end of each year in an equivalent investment plan. He stops making deposits after his 50th birthday. At that time, he leaves the balance in the account to collect interest until his 55th birthday.

- How much money will each person deposit in his investment plan?
- Who will have more money on his 55th birthday?

Solution

- Mike will contribute \$3000 per year for 20 years for a total of \$60 000, while Perry will contribute \$6000 per year for 20 years for a total of \$120 000, or twice the sum that Mike invested.

- (b) Mike makes annual deposits of \$3000 each for 20 years. First find the accumulated value on his 40th birthday. Substitute $R = 3000$, $i = 0.09$, and $n = 20$.

$$\begin{aligned} FV &= R \times \frac{(1+i)^n - 1}{i} \\ &= 3000 \times \frac{(1.09)^{20} - 1}{0.09} \\ &= 153\,480.36 \end{aligned}$$

Then the amount of the annuity collects compound interest on \$153 480.36 for 15 years, until Mike's 55th birthday. Substitute $P = 153\,480.36$, $i = 0.09$, and $n = 15$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 153\,480.36(1.09)^{15} \\ &= 559\,049.52 \end{aligned}$$

Mike's \$60 000 grows to \$559 049.52 by his 55th birthday.

Perry waits ten years and then makes annual deposits of \$6000 each for 20 years. Find the accumulated value on his 50th birthday. Substitute $R = 6000$, $i = 0.09$, and $n = 20$.

$$\begin{aligned} FV &= R \times \frac{(1+i)^n - 1}{i} \\ &= 6000 \times \frac{(1.09)^{20} - 1}{0.09} \\ &= 306\,960.72 \end{aligned}$$

Notice that the accumulated value in Perry's account is twice the accumulated in Mike's account, because each of Perry's payments was twice each of Mike's payments. However, Perry's balance of \$306 960.72 will collect compound interest for only five years. Substitute $P = 306\,960.72$, $i = 0.09$, and $n = 5$.

$$\begin{aligned} A &= P(1+i)^n \\ &= 306\,960.72(1.09)^5 \\ &= 472\,297.12 \end{aligned}$$

Perry's \$120 000 grows to \$472 297.12 by his 55th birthday.

Mike has accumulated more money on his 55th birthday, although Mike contributed half the amount that Perry contributed.

EXTENSION

Students should recognize the advantage of investing early. Suggest they read *The Wealthy Barber* by David Chilton and write a report, referring also to what they have learned in this chapter.

MANAGEMENT SUGGESTION

Arrange to have a financial planner as a guest speaker.

Practise, Apply, Solve 2.7

Questions with a Functions and Relations Focus:

4–10, 13–15, 17–23

Questions with a Functions Focus:

1–12, 14, 16, 18, 21, 22

- Finding the correct interest rate for each conversion period and the corresponding total number of conversion periods is critical for solving financial questions, regardless of the tool that students choose for calculating. No tool will work correctly without inputting the correct values, and marking assignments will be more difficult if students fail to learn this skill.

7. Knowledge and Understanding, p. 152

Management Suggestion

Students could work as partners to verify answers and provide feedback.

Achievement Chart Connection

Drawing a time line helps students to identify the four items—the deposit or payment, the interest rate, the term, and the compounding period—needed to solve any financial problem. An accurate time line indicates that students have grasped the concepts and correctly identified variables.

Level 3 Performance

In each time line, the arrows should point to the right, there should be marks for every month, and, at the end of each deposit (or payment) interval, the amount should be represented as $A = P(1+i)^n$. $R = 1600$, $i = 2.5\%$, and $n = 16$.

8. Before beginning any money problem, students should ask themselves the following questions: Are there one or more deposits or payments? Do I know, or want to know, the sum now, or later? Is the interest conversion period the same as the payment period? Is the annuity an ordinary annuity or an annuity due?

Assume an annuity is an ordinary annuity, unless otherwise stated.

- 9b Once students have learned section 2.11, they can quickly check the answers by setting **PMT** to **BEGIN**. The students must use the commands **seq** or **sum**, or use a spreadsheet.

14. Communication, p. 153

Management Suggestion

Working as partners, one student could “explain” to another student and this student could respond to the explanation. This dialogue could be taped.

Achievement Chart Connection

Students must have a thorough understanding of the development of the formulas and the meaning of the variables.

Level 3 Performance

Students should understand the role of the variables in the formulas and the development of the formulas. Students should also explain clearly, using appropriate terms, and could combine writing and mathematical expression in an essay.

18. Application, p. 154

Management Suggestion

Students should be able to answer this question independently. Perry’s case in the Additional Example: Functions is a similar problem.

Achievement Chart Connection

This problem involves two steps. Students need to distinguish between one “deposit” and many deposits and then select the appropriate formulas.

Level 3 Performance

Responses should include a time line and defined variables. Students must calculate the amount of an annuity and the accumulated amount of one “deposit.”

21. Thinking, Inquiry, Problem Solving, p. 154

Management Suggestion

Some students may prefer a theoretical solution and others a pragmatic one. Have students share these solutions. Teams can choose the best effort of the team members. Have students present a few of their solutions orally to the class, to broaden perspectives and to understand there may be several solutions to a problem. Seek consensus and discuss both options.

Achievement Chart Connection

Students are expected to create a plan, to apply the plan to both options (doubling the interest of annuity or doubling the term of an annuity), to show examples by assigning specific values, and to make a reasonable conclusion.

Level 3 Performance

Responses should show calculations, using either values or variables, for both options. Arguments should be consistent with the results.

22. Check Your Understanding, p. 154

Management Suggestion

Teams could discuss (a) for a few minutes and then participate in a whole class discussion. Their answers will be a strong indicator of their level of understanding. This is an opportunity for students to identify any misguided concepts. The solution to (b) is a direct application of the formula for future value. Students must solve for R .

The Chapter Problem Answers are on p. 2-67 of the *Teacher Resource*.

Challenge 1

Students could respond to this challenge at any time through research, but the answers will make more sense if they do this challenge when they do section 2.12.

When students research the Chapter Problem, remind them to read Challenge 1 and to research mortgages as well.

Challenge 2

The challenge will make sense after they have become familiar with the financial terminology, although section 1.8 provides the tools for calculating. Assign this challenge after section 2.8.

The Challenge Answers are on p. 2-82 of the *Teacher Resource*.