

2.8 Using Series to Analyze Financial Situations: Present Value

Specific Expectations

The following expectations are common to the Functions and Relations course and the Functions course.

- FA2.01 derive the formulas for compound interest and present value, the amount of an ordinary annuity, and the present value of an ordinary annuity, using the formulas for the n th term of a geometric sequence and the sum of the first n terms of a geometric series
- FA2.03 solve problems involving the amount and the present value of an ordinary annuity
- FA3.01 analyse the effects of changing the conditions in long-term savings plans (e.g., altering the frequency of deposits, the amount of deposit, the interest rate, the compounding period, or a combination of these) (Sample problem: Compare the results of making an annual deposit of \$1000 to an RRSP, beginning at age 20, with the results of making an annual deposit of \$3000, beginning at age 50)
- OC3.04 demonstrate the correct use of mathematical language, symbols, visuals (e.g., diagrams, graphs), and conventions

Evidence of Learning

Students should be able to

- design an informative time line for the present value of an annuity
- understand the connection between a geometric series and the present value of an annuity
- apply the present value formula for an ordinary simple annuity

Text Page 156

Time Required Functions and Relations or Functions: 225 min

Materials and Equipment

- scientific calculator

Overview

- Apply the sum of a geometric series to financial situations.
- Develop a formula for the present value of an ordinary simple annuity.
- Represent the present value of each payment with a time line.

Present Value

Students in the Functions and Relations course should be able to algebraically develop the formula for the present value of a ordinary simple annuity. They should recognize the connection between this formula and the formula for the sum of a geometric series.

Students will make this connection in step 3 of the *Think, Do, Discuss* in Part 2. Students who develop their skills in algebra are better able to check the reasonableness of answers produced using technology.

Management Options

Groups – Pairs may work through the *Think, Do, Discuss* of each part.

Teacher Demonstration – Less confident students might prefer a teacher-led demonstration.

Independent Work – For Part 1, students could practise “reading” a math text in a supportive environment. Answer the question of the first student who asks and then ask that student to help another classmate who has the same question. Reconvene as a class before proceeding to Part 2. Students may need assistance with steps 2 and 4 of the *Think, Do, Discuss* in Part 2.

Part 1: Representing the Present Value of an Ordinary Simple Annuity with a Series

Think, Do, Discuss, p. 156

MANAGEMENT SUGGESTION

This *Think, Do, Discuss* is also preparation for Challenge 2.

1. Is \$1000 required now to cover the cost of the \$1000 scholarship that will be awarded at the end of the first year? Explain.

Less than \$1000 is required now to cover the cost of the \$1000 scholarship that is to be awarded at the end of the year, because the investment earns interest during the year.

2. The interest rate is 9%/a, compounded annually. Determine an expression that would represent the amount that is needed now to provide the first \$1000 scholarship payment at the end of the first year. Express the present value of the payment as a product rather than as a quotient by using negative exponents.

$P = A(1 + i)^{-n}$, and $P = ?$, $A = 1000$,
 $i = 9\%$ or 0.09 , and $n = 1$

$$\begin{aligned} P_1 &= 1000(1 + 0.09)^{-1} \\ &= 1000(1.09)^{-1} \end{aligned}$$

3. Determine an expression for the present value of the second \$1000 scholarship. Express this present value as a product.

$P = A(1 + i)^{-n}$, and $P = ?$, $A = 1000$,
 $i = 9\%$ or 0.09 , and $n = 2$

$$P_2 = 1000(1.09)^{-2}$$

4. Repeat step 3 to find expressions for the present values of the third, fourth, and fifth payments.

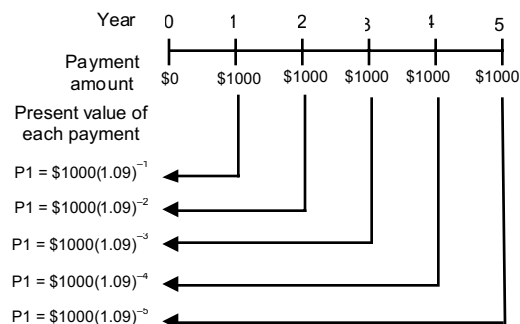
$P = A(1 + i)^{-n}$, and $P = ?$, $A = 1000$,
 $i = 9\%$ or 0.09 , and $n = 3, 4, 5$ for the third, fourth, and fifth payments, respectively

$$P_3 = 1000(1.09)^{-3}$$

$$P_4 = 1000(1.09)^{-4}$$

$$P_5 = 1000(1.09)^{-5}$$

5. Copy and complete the time line.



6. Use the expressions you found above, beginning with the present value of the fifth payment, P_5 , to write the series that represents the accumulated sum of all the present value payments. Do not find the value of each term.

MANAGEMENT SUGGESTION

Explain to students that it's possible to lose sight of the common ratio if the value of each term is calculated. Suggest students write each series in the following way to help see a pattern.

$$\begin{aligned} S_5 &= 1000(1.09)^{-5} + 1000(1.09)^{-4} \\ &\quad + 1000(1.09)^{-3} + 1000(1.09)^{-2} \\ &\quad + 1000(1.09)^{-1} \end{aligned}$$

7. What is the first term? What do you multiply the first term by to get the second term? What is this type of series? What is the common ratio? How many terms does the series have?

The first term, t_1 , is $1000(1.09)^{-5}$, which is also a . Multiply each successive term by the common ratio, r , $(1.09)^1$. This series is geometric and has five terms.

8. Use the formula you learned earlier in this chapter to find the sum of the present values of the payments. How much money would the club need to raise and then deposit in the trust account to sustain the scholarship over five years?

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1}, \text{ and } a = 1000(1.09)^{-5}, \\ r &= 1.09, \text{ and } n = 5 \text{ (the number of deposits)} \end{aligned}$$

$$S_5 = \frac{1000(1.09)^{-5}[(1.09)^5 - 1]}{(1.09) - 1}$$

$$= \frac{1000(0.649\,931\,386\,3)(0.538\,623\,954\,9)}{0.09}$$

$$= 3889.65$$

To sustain five \$1000 scholarships,
\$3889.65 must be deposited in the trust
account.

Part 2: Developing a Formula for the Present Value or Amount of an Ordinary Simple Annuity

Think, Do, Discuss, p. 157

1. What does i represent in the time line?
Explain why the exponent in the present value of the first payment is -1 . Explain why the exponent in the present value of the last payment is $-n$.

In the time line, i represents the interest rate per interest conversion period. The first payment earns interest for one interest period, so the exponent is 1. To get the present value, *divide* by the common ratio, so the exponent is -1 . The operation is division because the present value is less than the future value. The last payment earns interest for n periods, so the exponent is n . To get the present value of the last payment, *divide* by a multiple of the common ratio, so the exponent is $-n$.

2. Given Andrea's time line, write the series of terms, PV , that represents the present or discounted value of each payment of the annuity, beginning with the last payment. What type of series is PV ?

$$S_n = PV$$

$$PV = R(1+i)^{-n} + R(1+i)^{-n+1}$$

$$+ R(1+i)^{-n+2} + \dots + R(1+i)^{-2}$$

$$+ R(1+i)^{-1}$$

This series is geometric.

3. What is the first term of the series? What is the common ratio? Substitute these values in the formula you used in step 8 of Part 1.

$$a = R(1+i)^{-n}, r = (1+i)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{R(1+i)^{-n}[(1+i)^n - 1]}{(1+i) - 1}$$

4. You can simplify the equation in step 3, but before you do, what do you notice about the bases of the powers in this equation? What do you do to the exponents when you multiply these powers? Use the distributive property to simplify your equation. The result is the present value, PV , of an ordinary simple annuity.

The bases of the powers are the same. To multiply powers with the same base, keep the base and add the exponents.

$S_n = PV$ (Apply the distributive property.)

$$PV = \frac{R(1+i)^{-n}[(1+i)^n - 1]}{(1+i) - 1}$$

$$PV = \frac{R(1+i)^0 - R(1+i)^{-n}}{i}$$

$$PV = \frac{R[1 - (1+i)^{-n}]}{i}$$

$$\text{or } PV = R \times \frac{1 - (1+i)^{-n}}{i}$$

5. A math scholarship awards \$500 at the end of each year for ten years. Use the formula you developed in step 4 to find the present value that should be deposited in a trust account that pays 11%/a, compounded annually, to sustain the scholarship.

MANAGEMENT SUGGESTION

Teachers accustomed to using the S_n formula must be careful. In the formula for present value, $R \neq a$. R is the regular periodic payment of the annuity.

Substitute $R = 500$, $i = 11\%$ or 0.11, and $n = 10$ to find PV .

$$PV = R \times \frac{1 - (1+i)^{-n}}{i}$$

$$= 500 \times \frac{1 - (1.11)^{-10}}{0.11}$$

$$= 500 \times \frac{1 - 0.352\,184\,478\,8}{0.11}$$

$$= 2944.62$$

To sustain ten \$500 scholarships,
\$2944.62 should be deposited.

EXTENSION EXAMPLE 2

Ask Functions and Relations students to solve this example using the formula for the sum of a geometric series, as in Example 1 (c), Method 1, and discuss the advantages and disadvantages of each method.

Additional Example: Functions and Relations

Before you solve the following problem, consider Example 1, text page 159.

Chloe has just won the U Win Big Lottery. She must decide between two prize options: \$125 000 cash today or receive a \$1000 payment at the end of each month for 20 years. Chloe can expect a return of 9%/a, compounded monthly, if she invests the cash prize.

- (a) Find which option Chloe should choose by comparing the values for both options in i. 20 years and ii. 8 years.
 (b) Comment on the three methods, that is, the method in Example 1 on text page 159, the method in (a) i, and the method in (a) ii.

Solution

- (a) i. Option 1: Chloe could invest \$125 000 for 20 years.

Substitute $P = 125\,000$, $i = 9\% \div 12$ or 0.0075 , and $n = 12 \times 20$ or 240 to find A .

$$\begin{aligned} A &= P(1+i)^n \\ &= 125\,000(1.0075)^{240} \\ &\doteq 125\,000(6.009\,151\,524) \\ &= 751\,143.94 \end{aligned}$$

In 20 years, option 1 will have a value of \$751 143.94

Option 2 is an ordinary annuity such that $R = 1000$, $i = 0.0075$, and $n = 240$.

$$\begin{aligned} FV &= R \times \frac{(1+i)^n - 1}{i} \\ &= 1000 \times \frac{(1.0075)^{240} - 1}{0.0075} \\ &\doteq 1000 \times \frac{6.009\,151\,524 - 1}{0.0075} \\ &= 667\,886.87 \end{aligned}$$

In 20 years, option 2 will have a value of \$667 886.87. Option 1 has more value, as in Example 1.

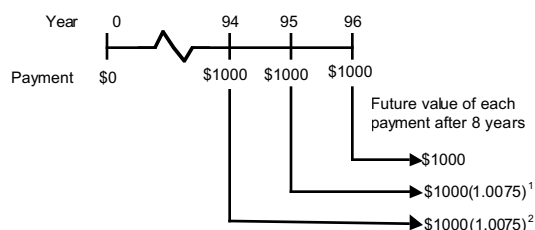
- (a) ii. Option 1: In 8 years, the \$125 000 would accumulate interest.

Substitute $P = 125\,000$, $i = 9\% \div 12$ or 0.0075 , and $n = 12 \times 8$ or 96 to find A .

$$\begin{aligned} A &= P(1+i)^n \\ &= 125\,000(1.0075)^{96} \\ &\doteq 125\,000(2.048\,921\,221) \\ &= 256\,115.15 \end{aligned}$$

In 8 years, option 1 will have a value of \$256 115.15.

Option 2 describes an ordinary annuity. Calculate the future value of the first 96 payments and the present value of the last 144 payments.

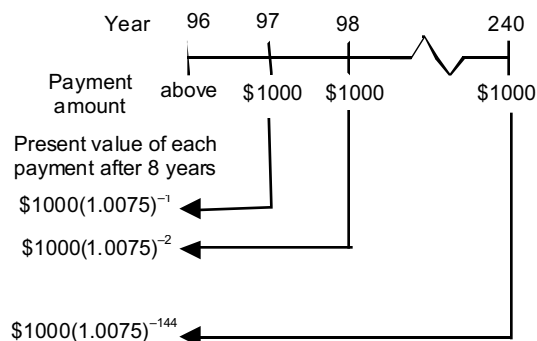


Substitute $R = 1000$, $i = 0.0075$, and $n = 96$ to find FV .

$$\begin{aligned} FV &= R \times \frac{(1+i)^n - 1}{i} \\ &= 1000 \times \frac{(1.0075)^{96} - 1}{0.0075} \\ &\doteq 1000 \times \frac{2.048\,921\,228 - 1}{0.0075} \\ &= 139\,856.16 \end{aligned}$$

After eight years, the payments could accumulate to \$139 856.16.

Now calculate the present value, the “present” being eight years from the start of the payments, of the rest of the payments.



Substitute $R = 1000$, $i = 0.0075$, $n = 240 - 96$ or 144 to find PV .

$$\begin{aligned}
 PV &= R \times \frac{1 - (1 + i)^{-n}}{i} \\
 &= 1000 \times \frac{1 - (1.0075)^{-144}}{0.0075} \\
 &\doteq 1000 \times \frac{1 - 0.340\,996\,810\,4}{0.0075} \\
 &= 87\,871.09
 \end{aligned}$$

In 8 years, the value of her remaining payments will be \$87 871.09: the total value of option 2 is \$227 727.26. The value of option 2 is, again, less than the value of option 1.

- (b) Whether you compare the options now, in 8 years, in 20 years, or at any other time, the value of option 1 will always be more than the value of option 2. When comparing methods, compare the values at a specific time. You know the present value of option 1, so calculate the present value of option 2. Therefore, the solution in the text is the most efficient. The solution in (a) ii is the least efficient.

Additional Example: Functions

Doug takes advantage of a 0% financing offer to purchase a new car, which costs \$24 000. He will repay the loan with monthly payments for five years. Fran wants a sportier model, but she must pay 6%/a, compounded monthly, interest. She can afford the same payments as Doug. How expensive a car can she finance?

Solution

Doug’s monthly payment for 5 years:

$$24\,000 \div (12 \times 5) = 400$$

Fran’s payments represent an annuity, where $R = 400$, $i = 6\% \div 12$ or 0.005, and $n = 12 \times 5$ or 60.

$$\begin{aligned}
 PV &= R \times \frac{1 - (1 + i)^{-n}}{i} \\
 &= 400 \times \frac{1 - (1.005)^{-60}}{0.005} \\
 &\doteq 400 \times \frac{1 - 0.741\,372\,196\,2}{0.005} \\
 &= 20\,690.22
 \end{aligned}$$

Fran could finance a car costing \$20 690.22.

Practise, Apply, Solve 2.8

Questions with a Functions and Relations Focus:

4–6, 9, 11, 12, 15, 17, 18, 20–23

Questions with a Functions Focus:

1–9, 11, 15, 17, 19–21

Management Suggestion

Before beginning any “money” problem, students should ask themselves the following questions: Are there one or more deposits or payments? Do I know, or want to know, the sum now, or later? Is the interest conversion period the same as the payment period? Is the annuity an ordinary annuity or an annuity due?

11. Knowledge and Understanding, p. 165

Management Suggestion

Students could work as partners to verify answers and provide feedback.

Achievement Chart Connection

Drawing a time line helps students to identify the four items—the deposit or payment, the interest rate, the term, and the compounding period—needed to solve any financial problem. An accurate time line indicates that students have grasped the concepts and correctly identified variables.

Level 3 Performance

In each time line, the arrows should point to the left, there should be a mark for every month, and, at the end of each deposit (or payment) interval, the amount should be represented as $P = A(1 + i)^{-n}$. $R = 40$, $i = 1.5\%$, and $n = 10$.

Students should accurately substitute in the present value formula and then calculate.

The down payment is added to the present value to calculate the cash price, and the interest charge is the difference between the payments and the amount financed.

15. Application, p. 165

Management Suggestion

Students could work in pairs, and each partner could solve one option. Then the partners can discuss their solutions and reach a conclusion. Each partner should have a complete solution to encourage clear expression. Students should discuss any advantages of either financing option.

Achievement Chart Connection

Students should draw a time line for each option. Students must identify the variables in each case and assume the same repayment period. Students should know that “paying cash” means obtaining financing from some place other than the dealership.

Level 3 Performance

Each time line should have left-pointing arrows, a mark at each month, and each payment should be represented by $A(1 + i)^{-n}$.

One time line should indicate that $PV = 32\,000$, $i = 0.2\%$, and $n = 60$.

The other time line should indicate that $PV = 29\,000$, $i = 0.45\%$, and $n = 60$.

Students should substitute correctly in the present value formula and then calculate accurately in both cases. Responses should include a reasonable conclusion based on calculations.

17. Communication, p. 166

Management Suggestion

Students, grouped in teams, should discuss possible solutions for a few minutes. If students need help getting started, suggest they begin with the present value formula.

Achievement Chart Connection

Students must plan a solution, gather information, and draw a reasonable conclusion.

Level 3 Performance

Responses may include an application of the present value formula, but solving this formula using algebra is difficult. Responses should include a comment recognizing that spreadsheet software allows the user to quickly test different values of i .

Students should integrate writing and mathematical expression in, for example, an essay, and communicate findings clearly and with justification.

20. Check Your Understanding, p. 166

Management Suggestion

Encourage students, working in teams, to create different cases, recognizing the limits of reasonable rates. Students could exchange solutions before finding the present values, (b), and they should appreciate the effects of changing the interest conversion period.

21. Thinking, Inquiry, Problem Solving, p. 166

Management Suggestion

Teams should verify their understanding of Carissa’s solution. They must order the steps, record them in a plan, assign tasks to teammates, and check each other’s calculations. Each team could present their solution, which would be an ideal situation for students to use the Learning Skills Self-Assessment Checklists. Provide the checklists to students before they begin. They may need a reminder if their cooperative skills begin to wane.

Achievement Chart Connection

Students are expected to analyze the information and sequence the calculations, using the appropriate variables. They will show understanding by viewing the solution from a different perspective.

Level 3 Performance

Responses should include a record of the sequence of tasks, a time line with properly assigned variables, and the series of payments. Students should apply the future value formula and the present value formula. Responses should also include a brief description of each step of the solutions for both parts in student friendly language.

The Chapter Problem Answers are on p. 2-67 of the *Teacher Resource*.

Challenge 1

Students could respond to this challenge at any time through research, but the answers will make more sense if they do this challenge when they do section 2.12.

When students research the Chapter Problem, remind them to read Challenge 1 and to research mortgages as well.

Challenge 2

The challenge will make sense after they have become familiar with financial terminology, although section 1.8 provides the tools for calculating. Assign this challenge now, provided that students understand that the scholarship is only interest and that the principal sum never diminishes.

The Challenge Answers are on p. 2-82 of the *Teacher Resource*.