

2.10 Equivalent Rates and General Annuities

Specific Expectations

The following expectations are common to the Functions and Relations course and the Functions course.

- FA3.01 analyse the effects of changing the conditions in long-term savings plans (e.g., altering the frequency of deposits, the amount of deposit, the interest rate, the compounding period, or a combination of these) (Sample problem: Compare the results of making an annual deposit of \$1000 to an RRSP, beginning at age 20, with the results of making an annual deposit of \$3000, beginning at age 50)
- OC3.02 present problems and their solutions to a group, and answer questions about the problems and the solutions
- OC3.05 use graphing technology effectively (e.g., use appropriate menus and algorithms; set the graph window to display the appropriate section of a curve)

Evidence of Learning

Students should be able to

- calculate interest rates compounded at any frequency that are equivalent to a rate compounded at any other frequency
- calculate the present value or future value of an ordinary general annuity or an ordinary annuity due

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Time Required Functions and Relations:

75 min

Time Required Functions: 110 min

Materials and Equipment

- TI-83 Plus calculator

Overview

- Distinguish between simple and general annuities.
- Calculate equivalent interest rates.
- Calculate the future value or the present value of an ordinary general annuity.

Equivalent Rates

It is possible to use the formula for the sum of a geometric series if the payment interval does not correspond to the interest conversion period, but calculations are awkward. In this section, students find the nominal interest rate that coincides with the payment interval. Then they may calculate the value of the annuity as if the annuity were an ordinary simple annuity.

MANAGEMENT SUGGESTION

Recommend to students that, when they are finding equivalent rates, to ask themselves “Should the equivalent rate be more or less than the given rate?” to check the reasonableness of an answer.

Management Options

Teacher Demonstration – For this section, direct teacher instruction is the most efficient approach.

MANAGEMENT SUGGESTION

Translating a sentence into a mathematical expression can help to clarify the rates and conversion periods. For example, “What is the nominal interest rate, compounded quarterly, that is equivalent to 12%/a, compounded monthly?” translates to $(1 + i)^4 = (1.01)^{12}$, where $i = 0.030\ 301$ per quarter means $4(0.030\ 301)$ or $0.121\ 204$ or $12.1204\%/a$, compounded annually.

TECHNOLOGY TIP

The solution in Example 4 (d) demonstrates how the TVM Solver may be used to solve problems involving equivalent rates.

Consolidate Your Understanding, p. 177

1. An investment earns 12%/a, compounded semiannually. Would the equivalent nominal rate, compounded quarterly, be greater than or less than 12%/a? Explain why.

The more frequently interest is compounded, the higher the interest earned will be. Compounding daily is better than weekly, which is better than monthly, which is better than quarterly, which is better than semiannually, which is better than annually. 12%/a, compounded semiannually, is greater than 12%/a. Definitely, 12%/a, compounded quarterly, is greater than 12%/a.

2. Write a problem so that the answer to the

$$\text{problem is } (1+i)^2 = \left(1 + \frac{0.09}{4}\right)^4.$$

Answers will vary, for example, “Find the interest rate compounded semiannually that is equivalent to 9%/a, compounded quarterly.”

3. Solve for i in question 2.

$$(1+i)^2 = \left(1 + \frac{0.09}{4}\right)^4$$

$$(1+i)^2 = (1.0225)^4 \quad \text{Take square root of both sides.}$$

$$1+i = (1.0225)^2$$

$$1+i \doteq 1.045506$$

$$i = 0.045506$$

The interest rate is 4.5506% per half year so that the interest rate equivalent to 9%/a, compounded quarterly, is 9.1012%/a, compounded semiannually.

Additional Example: Functions and Relations

A motorcycle shop has a used motorcycle on sale for \$4300. Your friend, Ludmilla, would like to buy the motorcycle. She could put \$400 down and finance the outstanding balance at 8.8%/a, compounded monthly, or at 9.1%/a, compounded annually. She can repay the loan by making equal weekly payments for 2.5 years or by making equal monthly payments for 3 years.

- (a) What would you advise Ludmilla to do so that the total cost of financing the motorcycle is minimized? Explain your advice.
- (b) Find the total cost of the motorcycle, based on your advice in (a).
- (c) You are convinced the purchase of the motorcycle is not in Ludmilla's best interest. You decide to show her how expensive it could be to finance the purchase. Describe how you can maximize the total cost of the motorcycle.
- (d) Find the total cost of the motorcycle based on your description for (c).

Solution

- (a) To minimize the cost, Ludmilla should

- make the \$400 down payment to reduce the amount borrowed
- make weekly payments to increase the frequency of payments and thereby reduce the principal faster
- choose the shortest repayment time, 2.5 years, to reduce the interest paid
- select 9.1%/a, compounded annually, because this interest rate is lower than 8.8%/a, compounded monthly

To decide which interest rate is lower, calculate the nominal interest rate, i , equivalent to 8.8%/a, compounded monthly.

$$(1+i)^1 = \left(1 + \frac{8.8\%}{12}\right)^{12}$$

$$1+i \doteq (1.007333)^{12}$$

$$1+i \doteq 1.091638$$

$$i = 0.091638$$

Therefore, 8.8%/a, compounded monthly, is equivalent to 9.1638%/a, compounded annually, which is more than 9.1%/a, compounded annually.

- (b) Based on the information in (a),

$$N = 2.5 \times 52 \text{ or } 130,$$

$I = 9.1$, because the interest rate is 9.1%/a, compounded annually

$$PV = 3900, \text{ because } 4300 - 400 = 3900$$

$$FV = 0$$

$P/Y = 52$, because the payments are weekly
 $C/Y = 1$, because the conversion period is a year

PMT:END, because a payment is made at the end of each month

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N=130
I%=9.1
PV=3900
PMT=-33.412454...
FV=0
P/Y=52
C/Y=1
PMT:END BEGIN

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Under these conditions, the weekly payment will be \$33.41. The total cost of the motorcycle will be the total of the weekly payments for 2.5 years and the down payment, $\$33.41(130) + \$400$ or \$4743.30.

- (c) To maximize the cost, select options opposite to those in (a). Do not make a down payment and choose monthly payments for 3 years, at the higher interest rate of 8.8%/a, compounded monthly.

- (d) Based on the information in (c),

$$N = 3 \times 12 \text{ or } 36$$

$I = 8.8$, because the interest rate is 8.8%/a, compounded monthly

$$PV = 4300$$

$$FV = 0$$

$P/Y = 12$, because the payments are monthly

$C/Y = 12$, because the conversion period is a month

PMT:END, because a payment is made at the end of each week

Each monthly payment will be \$136.34. The total cost of the motorcycle would be the monthly payments for 3 years, $\$136.34(36) = \$4908.24$.

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N=36
I%=8.8
PV=4300
PMT=-136.338947
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN

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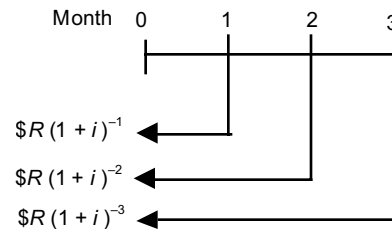
Additional Example: Functions

Yamir wants to buy a \$3400 laptop computer from a computer store. He can finance the purchase at a rate of 18%/a, compounded quarterly. Yamir plans to repay the purchase by making monthly payments for two years. Each payment will be made at the end of the month.

- (a) Draw a time line to represent the repayment of the loan.
 (b) Write the series that represents the present value of all the payments.
 (c) Determine Yamir's monthly payment.
 (d) Verify your solution using technology.

Solution

- (a)



Calculate the nominal interest rate, i , compounded monthly, that is equivalent to 18%/a, compounded quarterly.

$$(1+i)^{12} = \left(1 + \frac{18\%}{4}\right)^4$$

$$(1+i)^{12} = (1.045)^4$$

$$(1+i)^{12} \doteq 1.192\,518\,601$$

$$1+i \doteq 1.014\,780\,426$$

$$i = 0.014\,780\,426$$

Store the value for i in the calculator's memory for later use.

- (b) The series is $R(1.014\,780\,426)^{-1} + R(1.014\,780\,426)^{-2} + R(1.014\,780\,426)^{-3} + \dots + R(1.014\,780\,426)^{-24}$.

- (c) To find R , substitute $i = 0.014\,780\,426$, $n = 24$, and $PV = 3400$ into

$$PV = R \times \frac{1 - (1+i)^{-n}}{i}$$

$$3400 = R \times \frac{1 - (1.014780426)^{-24}}{0.014780426}$$

$$3400 = R(20.08157819)$$

$$R = 169.31$$

Yamir's monthly payment will be \$169.31.

- (d) There are 24 monthly payments, the interest rate is 18%/a, compounded quarterly, the present value is \$3400, the future value is 0, there are 12 payments each year, interest is compounded quarterly, and each payment is at the end of each month.

Practise, Apply, Solve 2.10

Questions with a Functions and Relations

Focus:

1–8, 10–17, 19, 21

Questions with a Functions Focus:

1–6, 8, 10–17, 19, 21

Management Suggestion

Before beginning any “money” problem, students should ask themselves the following questions: Are there one or more deposits or payments? Do I know, or want to know, the sum now, or later? Is the interest conversion period the same as the payment period? Is the annuity an ordinary annuity or an annuity due?

8. Knowledge and Understanding, p. 179

Management Suggestion

Individual partners can solve each part and then exchange solutions to check each other's work.

Achievement Chart Connection

Students apply their knowledge of equivalent rates and other concepts.

Level 3 Performance

Responses should include

- a time line, with arrows pointing to the right, a mark at every month, the interest rate, i , and appropriate variables
- a calculation of the equivalent weekly rate
- a series
- an accurate calculation of R in the formula for present value

10. Communication, p. 180

Management Suggestion

Partners can exchange the problems they created to describe the information in the TI-83 screens and check by creating a screen. If the screens do not agree, then partners should discuss how to clarify the information.

Achievement Chart Connection

Students are expected to interpret mathematical language that appears on a calculator screen.

Level 3 Performance

Students should justify and explain the problems they create.

The problem should correctly identify the unknown to be calculated, the number of payments, a reasonable interest rate, the correct interest conversion period, whether the annuity involves a future value or a present value, and whether the annuity is an annuity due or an ordinary annuity.

11. Application, p. 180

Management Suggestion

Suggest that students verify calculations using technology, but not to rely solely on technology, as it is possible to make entry errors.

Achievement Chart Connection

Students should find the equivalent monthly rate algebraically, confirm their calculations using technology, and compare the methods.

Level 3 Performance

Solutions should include

- a time line, where the arrows point left, the interest rate, i , and appropriate variables
- a calculation of the equivalent monthly rate
- a calculation of the monthly payment, using the present value formula and showing the correct identification of variables
- a copy or a sketch of the TI-83 Plus screen or a spreadsheet repayment schedule

19. Check Your Understanding, p. 181

Management Suggestion

A student can present the solution either in writing or orally to a group of students, or the teacher. The teacher can choose the audience from math, non-math, or younger math students. To communicate clearly, students must understand the concepts and use the math vocabulary in a student-friendly way.

21. Thinking, Inquiry, Problem Solving, p. 182

Management Suggestion

Students must provide concrete examples. Prepare cards that list a variety of options for interest rates, interest conversion periods, and repayment periods. Teams can select two cards, discuss strategies, assign individual tasks to team members, and suggest a scenario. They should select specific information on each card and solve both parts using this information. Half the team can work on (a), while the other half can work on (b). Then the groups can exchange the complete solutions for peer-editing and to check calculations. Suggest that their examples include more than one method, that is, for example, using the TVM Solver and using algebra. This activity would be ideal for self-assessment. Provide the Learning Skills Self-Assessment Checklists before students begin.

Achievement Chart Connection

Students create one or more models to test conjectures for (a) and (b). They must select the appropriate algebraic or technological tools and justify their conclusions.

Level 3 Performance

Responses should include for each part

- a record of sequenced tasks
- a list of variables
- a sketch of the results produced using technology, for example, a sketch of a TI-83 Plus screen showing the variables of the TVM Solver
- a proper algebraic solution
- a brief communication, that a peer could understand, describing each section
- integration of narrative and mathematical forms
- conclusions supported by evidence

LEARNING SKILLS ASSESSMENT

This task provides an ideal opportunity to observe all aspects of students' learning skills. Solving this question will take some time, which will allow students to develop confidence in problem-solving.

21. Add solutions to the collection for the student information forum.

The Chapter Problem Answers are on p. 2-67 of the *Teacher Resource*.

Challenge 1

Students could respond to this challenge at any time through research, but the answers will make more sense if they do this challenge when they do section 2.12.

When students research the Chapter Problem, remind them to read Challenge 1 and to research mortgages as well.

Challenge 2

The challenge will make sense after they have become familiar with the financial terminology, although section 1.8 provides the tools for calculating. If you did not assign this challenge in section 2.8, you could assign it now.

The Challenge Answers are on p. 2-82 of the *Teacher Resource*.