

2.1 Arithmetic Series

Specific Expectations

The following expectations are common to the Functions and Relations course and the Functions course.

- FA1.05 determine the sum of the terms of an arithmetic or a geometric series, using appropriate formulas and techniques

Evidence of Learning

Students should be able to

- generate an arithmetic series
- calculate the sum of an arithmetic series
- select the appropriate sum formula

Text Page 110

Time Required Functions and Relations or Functions: 75 min

Materials and Equipment

- scientific calculator

Overview

- Define an arithmetic series.
- Use Gauss's method to calculate the sum of an arithmetic series.
- Develop the formula for the sum of an arithmetic series.

Arithmetic Series

Students will solve problems involving the sum of an arithmetic sequence, or series. Functions and Relations students should be able to algebraically develop the formula for the sum of an arithmetic series by following the additional example provided here or by following the method suggested in question 22 of the *Practise, Apply, Solve*.

Management Options

Groups – Pairs or teams can do all the steps in the *Think, Do, Discuss*.

Teacher Demonstration – Lead Functions and Relations students through developing the formula for the sum of an arithmetic series. Show other students that Example 1 (b) works whether the terms in the sequence are added or either sum formula is applied (see *Key Ideas* in the text).

Independent Work – Allow individual students to practise “reading” a math text in a supportive environment. Circulate and ask questions to check understanding.

TECHNOLOGY SUGGESTION

Review section 2.6 now to facilitate checking answers.

Think, Do, Discuss, p. 110

1. Recall working with sequences in Chapter 1. What type of sequence describes the seating? Explain. How many seats are in the tenth row? Find the general term, t_n , that represents the number of seats in the n th row.

The sequence 12, 15, 18, 21, ... is an arithmetic sequence, because there is a common difference of 3 between each pair of successive terms. For $a = 12$, $d = 3$, and $n = 10$,

$$\begin{aligned} t_n &= a + (n - 1)d \\ t_{10} &= 12 + (10 - 1)(3) \\ &= 39 \end{aligned}$$

There are 39 seats in the tenth row.

$$\begin{aligned}
 t_n &= a + (n - 1)d \\
 &= 12 + (n - 1)(3) \\
 &= 12 + 3n - 3 \\
 &= 9 + 3n
 \end{aligned}$$

The general term is $t_n = 9 + 3n$, which represents the number of seats in the n th row.

2. How would you find the total number of seats in the first seven rows? Would this method be efficient for finding the number of seats in the first 30 rows? Explain.

Add $12 + 15 + 18 + \dots + t_7$. This method would be inefficient for finding the number of seats in the first 30 rows, because it is repetitive, tedious, and open to error.

3. The sum of the terms of a sequence is called a **series**. The sum of the first seven rows of seats is S_7 , where $S_7 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7$. The sum of the first 30 rows of seats is S_{30} . Express S_7 as the sum of seven terms, using the general term, t_n , you found in step 1. Write this sum in your notebook.

$$\begin{aligned}
 S_7 &= [9 + 3(1)] + [9 + 3(2)] + [9 + 3(3)] \\
 &+ [9 + 3(4)] + [9 + 3(5)] + [9 + 3(6)] \\
 &+ [9 + 3(7)]
 \end{aligned}$$

$$S_7 = 12 + 15 + 18 + 21 + 24 + 27 + 30$$

4. Use Gauss's method to guide you through the following steps.

- (a) Rewrite the sum for S_7 in reverse order so that t_7 is below t_1 , t_6 is below t_2 , and so on.

$$S_7 = 12 + 15 + 18 + 21 + 24 + 27 + 30$$

$$S_7 = 30 + 27 + 24 + 21 + 18 + 15 + 12$$

- (b) Add each pair of terms, for example, $t_7 + t_1$. What is the sum of each pair?

The sum of each pair is 42.

$$S_7 = 12 + 15 + 18 + 21 + 24 + 27 + 30$$

$$S_7 = 30 + 27 + 24 + 21 + 18 + 15 + 12$$

$$2S_7 = 42 + 42 + 42 + 42 + 42 + 42 + 42$$

- (c) How many pairs of terms are there? Calculate the sum of the pairs of terms and let this sum be equal to $2S_7$.

There are seven pairs of terms.

$$\begin{aligned}
 2S_7 &= 42(7) \\
 &= 294
 \end{aligned}$$

- (d) What must you do to $2S_7$ to determine S_7 , which is the number of seats in the first seven rows? Verify your solution by adding the terms in S_7 .

Divide each side by 2.

$$2S_7 = 294$$

$$\begin{aligned}
 S_7 &= 294 \div 2 \\
 &= 147
 \end{aligned}$$

$$\begin{aligned}
 S_7 &= 12 + 15 + 18 + 21 + 24 + 27 + 30 \\
 &= 147
 \end{aligned}$$

5. How many seats are in the 30th row? Use what you learned in the previous steps to find the total number of seats in the first 30 rows.

$$\begin{aligned}
 t_n &= 9 + 3n \\
 t_{30} &= 9 + 3(30) \\
 &= 99
 \end{aligned}$$

There are 99 seats in the 30th row.

Using Gauss's format, add the first and last terms, multiply by the number of terms, and divide by 2.

$$\begin{aligned}
 S_{30} &= (12 + 99)(30) \div 2 \\
 &= 1665
 \end{aligned}$$

There are 1665 seats in the first 30 rows.

6. Suppose you know the first term and the last term of an arithmetic sequence. Suggest a formula for finding S_n , the sum of any arithmetic series.

Using Gauss's method, add the first and last terms, multiply by the number of terms, and divide by 2.

$$S_n = [(t_1 + t_n)(n)] \div 2$$

MANAGEMENT SUGGESTION

It is good practice to list the sequence or series and state the value of the variables.

MANAGEMENT SUGGESTION

Provide an oral review of solving two equations with two unknowns. Elimination by adding or subtracting is appropriate when the series is arithmetic.

Additional Example: Functions and Relations

Develop two formulas for the sum of an arithmetic series. Start with the series $t_1 + t_2 + t_3 + \dots$.

Solution

$$S_n = t_1 + t_2 + t_3 + \dots + t_n \quad \text{①}$$

$$S_n = t_n + t_{n-1} + t_{n-2} + \dots + t_1 \quad \text{②}$$

$$2S_n = (t_1 + t_n) + \dots + (t_1 + t_n) \quad \text{①} + \text{②}$$

By Gauss's method, there are n sums of two terms on the right side of the equation.

$$2S_n = n(t_1 + t_n)$$

$$S_n = \frac{n(t_1 + t_n)}{2}$$

Use this formula when you know the first and last terms of a series.

Substitute $t_n = a + (n - 1)d$ and $t_1 = a$ in the sum formula above.

$$S_n = \frac{n[a + a + (n - 1)d]}{2}$$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Use this formula if you do not know the last term.

MANAGEMENT SUGGESTION

Students will be able to memorize these formulas through frequent use. Inexpensive prizes are usually ample incentive for memory work.

Additional Example: Functions

The third term of an arithmetic series is -3 and the sixth term is 15 . Calculate the sum of the first ten terms.

Solution

$$t_n = a + (n - 1)d$$

$$t_3: a + 2d = -3 \quad \text{①}$$

$$t_6: a + 5d = 15 \quad \text{②}$$

$$-3d = -18 \quad \text{①} - \text{②}$$

$$d = 3$$

Substitute $d = 3$ in ①.

$$a + 2(3) = -3$$

$$a = -15$$

The sequence is $-15, -9, -3, \dots$.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\begin{aligned} S_{10} &= \frac{10}{2}[2(-15) + (10 - 1)3] \\ &= 120 \end{aligned}$$

The sum of the first ten terms is 120 .

Practise, Apply, Solve 2.1**Questions with a Functions and Relations****Focus:**

1-3, 5-8, 10, 12-14, 18, 19, 21-24

Questions with a Functions Focus:

1-9, 11, 13, 15, 17, 18, 22

3. Communication, p. 114**Management Suggestion**

Let partners discuss the solution. Survey the class for ideas and then students can record the best ideas in their own words.

Achievement Chart Connection

The student's explanation should be clear and organized. The relatively new vocabulary should be used correctly. Students must justify each term in the expressions.

Level 3 Performance

Students should relate given information to the pattern, apply mathematical terms correctly, and understand essential concepts and terms.

7. Knowledge and Understanding, p. 114**Management Suggestion**

Discuss whether this question is similar to question 6 or to question 7 and which formula is appropriate in each case and why.

Achievement Chart Connection

Students must distinguish between S_{21} and t_{21} , then use the appropriate formula for the sum.

Level 3 Performance

Students should select the appropriate formula and state it correctly, assign variables correctly and substitute them in the formula, and calculate correctly.

13. Application, p. 15

Management Suggestion

Direct pairs to establish the variables, substitute values into a formula, and verify their results with other pairs in case of differences in answers.

Achievement Chart Connection

The context of this problem may be unfamiliar, because, in this sequence, the terms decrease.

Level 3 Performance

Students should

- create the correct series from the context of the problem
- select the appropriate formula and state it correctly
- assign variables correctly and substitute them in the formula
- calculate correctly

16. Students may use two separate series, one for rows 1 through 7 and another for rows 8 through 12.

17. To find the total time between the eighth second and the thirteenth second, calculate the difference between S_{13} and S_8 .

18. Check Your Understanding, p. 115

Management Suggestion

Students may record their explanations in a study journal. For work with annuities, students must understand the exact meaning of the variables, especially n . It is good practice for students to say the formula each time they answer in class.

22. Problem Solving and Inquiry, p. 116

Management Suggestion

The above Additional Example: Functions and Relations, shows an alternative solution. Expose Functions and Relations students to both proofs. Discuss both solutions in terms of simplicity.

Achievement Chart Connection

Students can duplicate the method in the *Think, Do, Discuss* with variable terms. The reasoning should be logical and consistent with the context.

Level 3 Performance

Students' solutions should be clearly organized and show logical reasoning that supports a conclusion. Students should recognize the pattern in the sequence.

The Chapter Problem—Financial Planning

Before students visit a local bank or other financial institution, discuss their plans in class so that the entire class does not visit the same local branch of a financial institution at the same time.

Some students may access the Internet outside of the classroom. There are many financial institutions that have Web sites. If resources are limited, ensure that answers are individual and that technical language is student friendly.

After students have completed section 2.5, they should be familiar with all terms in the Chapter Problem.

The Chapter Problem Answers are on p. 2-67 of the *Teacher Resource*.

Challenge 1

Students could respond to this challenge at any time through research, but the answers will make more sense if they do this challenge when they do section 2.12.

When students research the Chapter Problem, remind them to read Challenge 1 and to research mortgages as well.

Challenge 2

The challenge will make sense after they have become familiar with the financial terminology, although section 1.8 provides the tools for calculating. Assign this challenge after section 2.8.

The Challenge Answers are on p. 2-82 of the *Teacher Resource*.