

3.4 The Inverse Function

Specific Expectations

The following expectations are common to the Functions and Relations course and the Functions course.

- OC2.04 explain the relationship between a function and its inverse (i.e., symmetry of their graphs in the line $y = x$; the interchange of x and y in the equation of the function; the interchanges of the domain and range), using examples drawn from linear and quadratic functions, and from the functions $f(x) = \sqrt{x}$ and $f(x) = \frac{1}{x}$
- OC2.05 represent inverse functions, using function notation, where appropriate

Evidence of Learning

Students should be able to

- sketch the inverse of a function given the graph of the function
- write the equation of the inverse given the equation of the function
- list points of the inverse relation given, or knowing, points of the original function
- identify the domain and range of a function and its inverse
- test a sequence to determine whether it is arithmetic or not by finding the differences between successive terms

Text Page 247

Time Required Functions and Relations: 120 min

Time Required Functions: 150 min

Materials and Equipment

- scientific calculator, graph paper

Overview

- Algebraically manipulate the equation of a function to find the equation of its inverse.
- Understand how the domain, the range, and the graph of a function changes for the inverse relation.

MANAGEMENT SUGGESTION

Direct students to methodically organize operations for the function and its inverse. For example, for the function $f(x) = x^2 - 2$, graph the function, identify the domain and range, find and graph the inverse, identify the domain and range of the inverse, and discuss the fact that the inverse is not a function in this case.

Management Options

Groups – Invite groups of students to work through the questions of the *Think, Do, Discuss* of Part 1. Some students may not recognize the meaning of the symbol $\in$.

When students have completed the *Think, Do, Discuss*, discuss as a class how to “reverse” operations before proceeding. Because notation must be carefully written, you might want to lead the second part of this lesson.

Teacher Demonstration – Show how to solve for the inverse relation algebraically, demonstrating the expected mathematical form. Give students several examples of functions and let them solve for the inverse relation for each function on their own. Give them immediate feedback after each example, if needed. Present a function using the overhead, allow time for students to solve for the inverse relation, and then uncover the solution. After students have worked through several examples, collect samples of students’ work for assessment.

EXTENSION

Ask students to research the meaning of a one-to-one relation.

Part 1: Defining the Inverse Function

Think, Do, Discuss, text page 247

1. The formula for converting a temperature in degrees Celsius to degrees Fahrenheit is

$$F = \frac{9}{5}C + 32.$$

An American visitor to

Canada uses this simpler rule to convert from Celsius to Fahrenheit: double the Celsius temperature, then add 30.

- (a) Copy and complete the table using the visitor's rule.

Temperature (°C)	Temperature (°F)
10	50
15	60
20	70
25	80
30	90

- (b) What is the independent variable? the dependent variable?

The temperature in degrees Celsius is the independent variable and the temperature in degrees Fahrenheit is the dependent variable.

- (c) Does this rule define a function? Explain.

This rule defines a function, because the rule predicts a unique Fahrenheit temperature for each Celsius temperature.

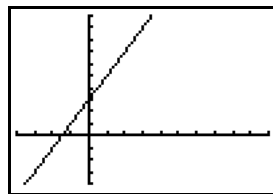
- (d) Let f represent the rule. What ordered pair, $(0, \blacksquare)$, belongs to f ?

$(0, 30)$

- (e) Let x represent the temperature in degrees Celsius. Write the equation for this rule in function notation.

$$f(x) = 2x + 30$$

- (f) Graph the relation. Use the same scale of -40 to 100 on each axis.



- (g) In the table, $f(10) = 50$, which corresponds to a point on the graph of $y = f(x)$. What is the x -coordinate of this point? What is its y -coordinate?

The x -coordinate is 10 and the y -coordinate is 50.

2. A Canadian visited Florida and used this rule to convert the temperature from degrees Fahrenheit into degrees Celsius. To convert 50°F into a temperature in degrees Celsius, the Canadian subtracted 30 and divided the result by 2 to get 10°C .

- (a) Copy and complete the table using this rule.

Temperature (°F)	Temperature (°C)
50	10
60	15
70	20
80	25
90	30

- (b) What is the independent variable? the dependent variable?

The independent variable is the temperature in degrees Fahrenheit and the dependent variable is the temperature in degrees Celsius.

- (c) This relationship is called the **inverse** of the function in step 1. The first function converts from degrees Celsius into degrees Fahrenheit. The inverse function is the “reverse” of the original function because it converts from degrees Fahrenheit into degrees Celsius. Compare the tables in steps 1(a) and 2(a). Describe how you could use a table for a relation to get a table for its inverse relation.

Interchange the columns of the table for the relation to get the table for the inverse relation.

- (d) In mathematics, f is often used to denote the original function, and f^{-1} is used to denote the **inverse** function. Notice that $(10, 50) \in f$ and $(50, 10) \in f^{-1}$. Describe how you can find an ordered pair that belongs to the inverse function if you know an ordered pair that belongs to the original function.

To find the ordered pair for the inverse function, interchange the x - and y -values of the ordered pair of the original function.

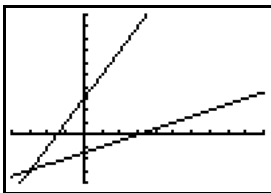
- (e) What operations will reverse or “undo” the original rule? Write the rule that the Canadian could use to convert temperature from degrees Fahrenheit into degrees Celsius.

Subtract 30 from the temperature in degrees Fahrenheit and then divide by 2 to get the temperature in degrees Celsius.

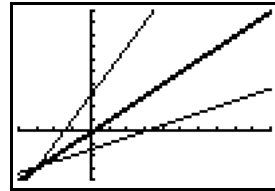
- (f) Let x represent the temperature in degrees Fahrenheit. Write the equation for this rule in function notation.

$$f^{-1} = \frac{x - 30}{2}$$

- (g) Graph the inverse relation on the same axes you drew in step 1(f).



- (h) Draw the line with equation $y = x$ on the same axes. Fold your graph paper along the line $y = x$. What do you notice about the graphs of f and f^{-1} ?



When the graph paper is folded along the line $y = x$, the graphs of the function and the inverse match.

- (i) In the table, $f^{-1}(50) = 10$, which corresponds to a point on the graph of $y = f^{-1}(x)$. What is the x -coordinate of this point? What is its y -coordinate? What is the corresponding point on the graph of $y = f(x)$?

The x -coordinate is 50. The y -coordinate is 10. The point on the graph of $y = f(x)$ is $(10, 50)$.

- (j) What are the coordinates of the point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$? What is the significance of this point?

The intersection point is $(-30, -30)$. This is the point where both lines also intersect the line of reflection, $y = x$.

Part 2: Determining an Algebraic Expression for the Inverse Function

Consolidate Your Understanding, p. 251

- How can you find the equation of the inverse relation if you know the equation of the relation?

Switch the positions of x and y in the equation of the relation. Then solve for y .

2. How is finding $f^{-1}(x)$ different from finding the inverse of a relation?

The expression $f^{-1}(x)$ represents the inverse function of the original function $f(x)$ and it also represents the value of the function for any value of x . There is little difference between finding $f^{-1}(x)$ and finding the inverse because they are very similar and to find both you interchange x and y and then solve for y . The inverse of a relation though is not necessarily a function.

3. Describe how you would find $f^{-1}(x)$ if you were given $f(x)$.
- Express function $f(x)$ as an equation in slope-intercept form.
 - Interchange x and y .
 - Isolate y to create a new equation in the slope-intercept form. In other words, solve for y .
 - Replace y with $f^{-1}(x)$.

Additional Example: Functions and Relations

Explain why $(f^{-1})^{-1}(x) = f(x)$.

Solution

The notation $(f^{-1})^{-1}$ means the inverse of the inverse of f . If we interchange the x - and y -values of f to get f^{-1} and then interchange those values again, the result will be the original set of ordered pairs. Therefore, $(f^{-1})^{-1} = f$.

The notation $(f^{-1})^{-1}(x)$ denotes the function value as an expression in x . Since $(f^{-1})^{-1} = f$, their function values will be identical.

Additional Example: Functions

A bank agrees to exchange your Canadian money at \$1 CDN buys \$0.63 U.S. It also advertises that it will convert U.S. money at \$1 U.S. buys \$1.57 CDN. Let f represent the function that converts Canadian money to U.S. and g represent the function that converts U.S. money to Canadian money. Determine expressions for f , f^{-1} , g , and g^{-1} . Compare the functions to determine if the bank makes a profit on any of the exchanges.

Solution

To convert Canadian currency to U.S. currency, multiply the input value by 0.63. Therefore, $f(x) = 0.63x$. To reverse the process, divide by 0.63, so $f^{-1}(x) = \frac{x}{0.63}$.

To convert U.S. currency to Canadian currency, multiply the input value by 1.57. Therefore, $g(x) = 1.57x$. To reverse the process, divide by 1.57, so $g^{-1}(x) = \frac{x}{1.57}$.

$$\begin{aligned} f^{-1}(x) &= \frac{x}{0.63} \\ &= \frac{1}{0.63}x \\ &\doteq 1.587x \end{aligned}$$

In other words, the bank should pay \$1.587 Canadian for each U.S. dollar, but it only pays \$1.57. The bank makes a profit of 1.7¢ per dollar on each transaction.

$$\begin{aligned} g^{-1}(x) &= \frac{x}{1.57} \\ &= \frac{1}{1.57}x \\ &\doteq 0.637x \end{aligned}$$

In other words, the bank should pay \$0.637 U.S. for each Canadian dollar, but it only pays \$0.630. The bank makes a profit of 0.7¢ per dollar on each transaction.

In each case, the bank makes a slight profit on the exchange. The real exchange rate is probably between these two values.

ASSESSMENT SUGGESTION

Communication, Knowledge And Understanding

Create a table with these headings: algebraic expression for $f(x)$, graph of $f(x)$, algebraic expression for inverse, graph of inverse. Present several examples of functions to students. For the examples, fill in one column and ask students to fill in the blanks.

Practise, Apply, Solve 3.4

Questions with a University Focus:

1–6, 8, 9, 12–14, 16–22

Questions with a University/College Focus:

1–7, 9–11, 13, 16, 17, 20, 22

9. Knowledge and Understanding, p. 257

Management Suggestion

Collect and assess responses. Provide opportunities for students to show improved understanding at a future time, where required.

Achievement Chart Connection

Students must demonstrate understanding of many characteristics of the inverses of functions. They might create graphs by hand or with technology.

Level 3 Performance

Students should correctly interchange variables to find the inverse of the function. They should identify the values of the dependent and independent variables for the function and its inverse. Students must recognize $y = x$ as a line of reflection and find the point of intersection between the function, the inverse, and $y = x$.

13. Communication, p. 258

Management Suggestion

Ask partners to exchange their responses with each other and then request volunteers to share a few responses with the class.

Achievement Chart Connection

Students are expected to use mathematical language, symbols and conventions in their descriptions and to justify the situation they choose as an example of the use of an inverse.

Level 3 Performance

Responses should include an efficient, accurate creation of the inverse and a clear description of a reasonable situation where an inverse might be used.

17. Thinking, Inquiry, Problem Solving, p. 259

Management Suggestion

While groups of students work on this or a similar question, you might confer with one or two groups and listen to the group members' ideas and approaches to the problem.

Achievement Chart Connection

Students represent the situation in a mathematical (algebraic, graphical) form, make and interpret conclusions, and reflect on the reasonableness of their results.

Level 3 Performance

Students must use technology effectively to create scatter plots and determine the line of best fit. They should manipulate the equation (algebraic model) to determine unknown quantities and make predictions. Students should correct unreasonable results.

20. Application, p. 260

Management Suggestion

Discuss the possible reasons for choosing to encode information such as credit card numbers.

Achievement Chart Connection

Students use functions and inverses in an unfamiliar situation.

Level 3 Performance

Students must use a function instead of just algebra and work independently.

22. Check Your Understanding, p. 260

Management Suggestion

Give students a few minutes to jot down their responses and then invite volunteers to share their ideas in a class discussion, allowing students to confirm their understanding of the concepts.

ASSESSMENT SUGGESTION

Organization

Students might prepare a flow chart showing the steps required to find the inverse of a given function. This flow chart might become part of a chapter summary or a larger course summary.

The Chapter Problem—Cryptography

You might want to encourage students to use graphing calculators or spreadsheets to graph the original relationship and its inverse. It might also be useful to reinforce the connections of the graphs to the input/output diagrams for both the encoder and decoder. Without formally being defined, the concept of a one-to-one function is developed through this activity.

The Chapter Problem Answers are on p. 3-45 of the *Teacher Resource*.

Challenge 1

Remind students that they should be able to

- create a table showing the relationship between the original alphabet and the possible coded letters based on frequency.
- use the “draft” function to create the table for the inverse and then decode the message. Students will have to adjust the coding and decoding functions to decode the message.

Challenge 2

Suggest students think carefully about the domain and range of the coded function of the decoder. Note that the decoder is not the inverse of the coding function.

The Challenge Answers are on p. 3-48 of the *Teacher Resource*.