

3.6 Investigating Properties of Inverse Functions

Specific Expectations

These expectations are common to the Functions and Relations course and the Functions course.

- OC2.04 explain the relationship between a function and its inverse (i.e., symmetry of their graphs in the line $y = x$; the interchange of x and y in the equation of the function; the interchanges of the domain and range), using examples drawn from linear and quadratic functions, and from the functions $f(x) = \sqrt{x}$ and $f(x) = \frac{1}{x}$
- OC2.05 represent inverse functions, using function notation, where appropriate
- OC2.06 represent transformations (e.g., translations, reflections, stretches) of the functions defined by $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \sin x$, and $f(x) = \cos x$, using function notation

Evidence of Learning

Students should be able to

- find the equation for the inverse of a given function
- express an inverse using function notation
- determine the domain and range for a function and its inverse
- determine whether an inverse is a function or not, and explain why

Text Page 265

Time Required Functions and Relations or Functions: 150 min

Materials and Equipment

- TI-83 Plus calculator

Overview

- Learn that the inverse of a function may not be a function.
- Investigate the domain and range of a function and its inverse and learn how to restrict the domain and range so that the inverse of the function is also a function.
- Use $f(x)$ and $f^{-1}(x)$ notation appropriately.

MANAGEMENT SUGGESTION

Review students' prior knowledge of functions, domain, range, $f(x)$ notation, and the vertical line test.

Management Options

Groups – Give students in small groups some time to work through the *Think, Do, Discuss* of each part. Also discuss and check the responses and answer any questions.

Teacher-Led Discussion – Use the *Think, Do, Discuss* of each part or a similar plan to lead students to an understanding of the inverses of the specific functions. Stress the interchanging of variables in the equation of each function, and point to the domain, range, graph, and table for each specific function and its inverse.

Independent Work – In some classes, you may expect students to work through the questions on their own and come to class prepared to discuss the ideas.

EXTENSION

Ask students to consider these questions:

1. What examples of inverse relations have you met in previous mathematics classes?
2. Explain the meaning of $f^{-1}(x)$, $f(x^{-1})$, $\frac{1}{f(x)}$ as related to $f(x)$.

Part 1: Investigating the Relationships Between

$$y = \sqrt{x} \text{ and } y = x^2$$

Emphasize the interchanging of x and y .

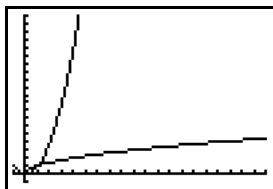
Think, Do, Discuss, text page 266

1. (a) Copy and complete the tables. List the coordinates of points that belong to $y = \sqrt{x}$ in the left table. Use values for x that include 0. List the corresponding points, with the coordinates reversed, for $y = x^2$ in the right table.

x	$y = \sqrt{x}$
0	0
1	1
4	2
9	3
16	4

x	$y = x^2$
0	0
1	1
2	4
3	9
4	16

- (b) Graph $y = \sqrt{x}$ and its inverse using the same colour. Use the same scale for each axis, if possible.



2. (a) Copy and complete the tables. List the coordinates of points that belong to $y = -\sqrt{x}$ in the left table. Use values for x that include 0. List the corresponding points, with the coordinates reversed, for $y = x^2$ in the right table.

x	$y = -\sqrt{x}$
0	0
1	-1
4	-2
9	-3
16	-4

x	$y = x^2$
0	0
-1	1
-2	4
-3	9
-4	16

- (b) Graph $y = -\sqrt{x}$ and its inverse using the same colour on the same axes as in step 1(b).

(See the graph in step 3.)

3. Graph $y = x$ on the same axes using a different colour. The graph should look like this one.

The red part of $y = x^2$ is called the **right branch** of the parabola, while the blue section is the **left branch**.

4. Fold the graph paper along the line $y = x$, and hold the paper up to the light.

- (a) Which part of the graph is the inverse of $y = \sqrt{x}$? Explain.

The right branch of $y = x^2$ is the inverse of $y = \sqrt{x}$. The coordinates of each point have been interchanged. The graphs have only positive values.

- (b) Which part of the graph is the inverse of $y = -\sqrt{x}$? Explain.

The right branch of $y = x^2$ is the inverse of $y = -\sqrt{x}$. The coordinates of each point have been interchanged. For the original function, the y -values are negative, and this portion of the curve is below the x -axis. For the inverse function, the x -values are negative, so this portion of the curve is left of the y -axis.

- (c) Which part of the graph is the inverse of $y = x^2$? Explain.

The upper and lower branches of the parabola that opens to the right represent the inverse of $y = x^2$. The inverse relation is not a function.

5. State the domain and range of each function. Explain your findings.

(a) $y = x^2$

The domain is $\mathbf{R}$. The range is $\{y \mid y \geq 0, y \in \mathbf{R}\}$.

(b) $y = \sqrt{x}$

The domain is $\{x \mid x \geq 0, x \in \mathbf{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathbf{R}\}$.

(c) $y = -\sqrt{x}$

The domain is $\{x \mid x \geq 0, x \in \mathbf{R}\}$. The range is $\{y \mid y \leq 0, y \in \mathbf{R}\}$.

6. (a) Does $y = x^2$ define a function? Explain.

Yes, because each x -value yields only one y -value.

- (b) Does $y = \sqrt{x}$ define a function? Explain.

Yes, because each x -value yields only one y -value.

- (c) Does $y = -\sqrt{x}$ define a function? Explain.

Yes, because each x -value yields only one y -value.

- (d) Is the inverse of $y = x^2$ a function? Explain.

The inverse is not a function. When the variables are interchanged, the result is

$$x = y^2$$

$$y = \pm\sqrt{x}$$

A vertical line will intersect the graph in at least two places.

- (e) Is the inverse of $y = \sqrt{x}$ a function? Explain.

Yes, because each x -value yields only one y -value.

- (f) Is the inverse of $y = -\sqrt{x}$ a function? Explain.

Yes, because each x -value yields only one y -value.

7. (a) What restriction could you place on the domain of $y = x^2$ so that its inverse is $y = \sqrt{x}$?

$$x \geq 0$$

- (b) What restriction could you place on the domain of $y = x^2$ so that its inverse is $y = -\sqrt{x}$?

$$x \leq 0$$

- (c) What is the equation of the inverse of $y = x^2$?

$$x = y^2 \text{ or } y = \pm\sqrt{x}$$

- (d) What restriction could you place on the range of the inverse of $y = x^2$ so that it would be a function, without changing its domain? Explain.

One possible restriction of the range of the inverse is $y \leq 0$. This restriction would eliminate the top (positive) branch of the parabola. Then each value for the domain of the inverse would correspond to a single value for the range.

- (e) Is there another restriction that you could place on the range of the inverse of $y = x^2$ so that it would be a function, without changing its domain? Explain.

Another restriction of the range of the inverse is $y \geq 0$. This restriction would eliminate the bottom (negative) branch of the parabola. Then each value for the domain of the inverse would correspond to a single value for the range.

Part 2: Restricting the Domain of a Quadratic Function So That the Inverse Is a Function

Think, Do, Discuss, text page 267

1. (a) Set your calculator to degree mode. Then evaluate each of the following.

- i. $\sin 30^\circ$ ii. $\sin 150^\circ$
iii. $\sin 390^\circ$ iv. $\sin 510^\circ$

The value is 0.5 for each case.

- (b) For $f(x) = \sin(x)$, all of the ordered pairs $(30^\circ, \blacksquare)$, $(150^\circ, \blacksquare)$, $(390^\circ, \blacksquare)$, and $(510^\circ, \blacksquare)$ belong to f .

$(30^\circ, 0.5)$, $(150^\circ, 0.5)$, $(390^\circ, 0.5)$, and $(510^\circ, 0.5)$

- (c) Use your answers from (a) to list ordered pairs that belong to f^{-1} .

$(0.5, 30^\circ)$, $(0.5, 150^\circ)$, $(0.5, 390^\circ)$, and $(0.5, 510^\circ)$

- (d) What is the value of $f^{-1}(0.5)$?

There are at least four correct but different answers, 30° , 150° , 390° , and 510° .

- (e) f^{-1} should be the inverse of sine. Use your calculator to evaluate $\sin^{-1}(0.5)$.
 $\sin^{-1}(0.5) = 30^\circ$

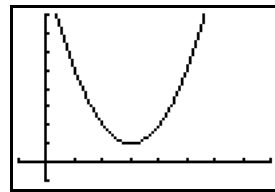
- (f) Explain what the calculator has calculated.

The calculator has given the smallest positive measure, and, at this point, $\sin^{-1} x$ is a function for a specific part of the curve.

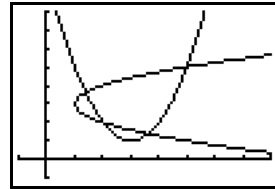
MANAGEMENT SUGGESTION

You may want to sketch the graph of $y = \sin^{-1} x$ and point to the answer for (e). Show that this relation is a function only over certain intervals of the range.

2. (a) Sketch any parabola that has vertex $(3, 1)$ and opens up.



- (b) Sketch its inverse.



- (c) How much of the graph of the inverse would you have to erase so that it would pass the vertical line test?

Either branch of the inverse should be erased.

- (d) Redraw the graph of the inverse so that it passes the vertical line test. Stating this inequality places a restriction on the inverse relation to ensure it is a function.

$$\{y \mid y \leq 3, y \in \mathbf{R}\} \text{ or } \{y \mid y \geq 3, y \in \mathbf{R}\}$$

- (e) How would you rewrite the domain of the original function so that its inverse passes the vertical line test? Stating the new domain with a restriction on the original function guarantees that the inverse is a function.

$$\{x \mid x \leq 3, x \in \mathbf{R}\} \text{ or } \{x \mid x \geq 3, x \in \mathbf{R}\}$$

- (f) State the new domain and range of the function and its inverse.

For the new (restricted) function, or the right branch of the original function, the domain is $\{x \mid x \geq 3, x \in \mathbf{R}\}$ and the range is $\{y \mid y \geq 1, y \in \mathbf{R}\}$. For the inverse, the domain is $\{x \mid x \geq 1, x \in \mathbf{R}\}$ and the range is $\{y \mid y \geq 3, y \in \mathbf{R}\}$.

For the new (restricted) function, or the left branch of the original function, the domain is $\{x \mid x \leq 3, x \in \mathbf{R}\}$ and the range is $\{y \mid y \geq 1, y \in \mathbf{R}\}$. For the inverse, the domain is $\{x \mid x \geq 1, x \in \mathbf{R}\}$ and the range is $\{y \mid y \leq 3, y \in \mathbf{R}\}$.

- (g) Why does the vertex help you to see an appropriate restriction?

The vertex is the point at which the parabola can be divided into two, and this point represents the maximum or minimum values for the domain or range.

- (h) What other restriction could you place on the domain of the parabola?

To make the inverse a function, we could restrict the domain more than is necessary. For example, for domain $\{x \mid x \geq 4, x \in \mathbf{R}\}$, the inverse would be a function, but more than is necessary of the inverse is eliminated.

3. For $y = 3(x + 1)^2 + 4$,

- (a) state the coordinates of the vertex of this parabola

$(-1, 4)$

- (b) describe its location and direction of the opening

The parabola has vertex $(-1, 4)$, opens up, and passes through $(0, 7)$.

- (c) what restriction could you place on the domain of the function to ensure that the inverse is also a function?

$x \leq -1$ or $x \geq -1$

4. (a) What values of y satisfy $y^2 = 16$?

$y = 4$ or $y = -4$

- (b) Solve $y^2 = 7$.

$y = \pm\sqrt{7}$

- (c) Solve $y^2 = x$ for y .

$y = \pm\sqrt{x}$

- (d) The equation of the inverse of $f(x) = x^2$ is $x = y^2$. Determine $f^{-1}(x)$. Why is the inverse not a function?

$f^{-1}(x) = \pm\sqrt{x}$. The inverse is not a function, because any value of x creates two different values for y .

- (e) What restriction could you place on the domain of f so that the inverse is a function? In this case, what is $f^{-1}(x)$?

A possible restriction for f is $x \geq 0$. In this case, $f^{-1}(x) = \sqrt{x}$.

5. (a) Solve $y^2 = 25$.

$y = \pm 5$

- (b) Solve $(y + 1)^2 = 25$.

$(y + 1)^2 = 25$

$y + 1 = \pm 5$

$y = -1 + 5$ or $y = -1 - 5$

$y = 4$ or $y = -6$

- (c) Solve $2(y + 1)^2 = 50$.

$2(y + 1)^2 = 50$

$(y + 1)^2 = 25$

$y + 1 = \pm 5$

$y = 5 - 1$ or $y = -5 - 1$

$y = 4$ or $y = -6$

- (d) Solve $2(y + 1)^2 - 3 = 47$.

$2(y + 1)^2 - 3 = 47$

$2(y + 1)^2 = 50$

$(y + 1)^2 = 25$

$y + 1 = \pm 5$

$y = 5 - 1$ or $y = -5 - 1$

$y = 4$ or $y = -6$

(e) Solve $2(y+1)^2 - 3 = x$ for y .

$$\begin{aligned} 2(y+1)^2 - 3 &= x \\ 2(y+1)^2 &= x+3 \\ (y+1)^2 &= \frac{x+3}{2} \\ y+1 &= \pm \sqrt{\frac{x+3}{2}} \\ y &= \pm \sqrt{\frac{x+3}{2}} - 1 \end{aligned}$$

(f) Determine $f^{-1}(x)$ if $f(x) = 2(x+1)^2 - 3$.

$$f^{-1}(x) = \pm \sqrt{\frac{x+3}{2}} - 1$$

6. (a) Solve $\sqrt{y} = 3$.

$$y = 9$$

(b) Why does $\sqrt{y} = -3$ have no solution?

To solve, you would square both sides of the equation. No squared number will produce a negative number.

(c) What must be true about x if $\sqrt{y} = x$?

$$x \geq 0$$

(d) Solve for y if $\sqrt{y} = x$.

$$y = x^2, x \geq 0$$

(e) The inverse of $\sqrt{y} = x$ is $x = \sqrt{y}$.

Solve for y .

$$y = x^2, x \geq 0$$

Additional Example: Functions and Relations

Use $f(x) = x^3 + 4$ or use an example of your own to show that $g^{-1}(x) = f(x)$ if $g(x) = f^{-1}(x)$.

Solution

$$\begin{aligned} f(x) &= x^3 + 4 \\ y &= x^3 + 4 \end{aligned}$$

The inverse is then

$$\begin{aligned} x &= y^3 + 4 \\ y &= \sqrt[3]{x-4} \\ f^{-1}(x) &= \sqrt[3]{x-4} \end{aligned}$$

Let the original function be $g(x) = \sqrt[3]{x-4}$.

The inverse is then

$$\begin{aligned} x &= \sqrt[3]{y-4} \\ x^3 &= y-4 \\ y &= x^3 + 4 \end{aligned}$$

And $g^{-1}(x) = x^3 + 4$, which is the same as $f(x)$.

Additional Example: Functions

Explain how you could use the graph of a function to predict whether the inverse of the function is also a function or not.

Solution

The inverse is not a function if you can draw a horizontal line on the graph so that the line intersects two or more points.

ASSESSMENT SUGGESTION

Application, Communication

Collect and assess students' responses to this question.

Using appropriate mathematical symbols and vocabulary, and referring to a specific function, explain how to find the inverse of a function. Note the effects on the domain and range of the function and also its inverse.

Practise, Apply, Solve 3.6

Questions with a University Focus:

1–6, 9–20

Questions with a University/College Focus:

1–6, 9–14, 15, 17–19

10. Knowledge and Understanding, p. 275 Management Suggestion

Collect and assess the responses, and note the following Assessment Suggestion.

Achievement Chart Connection

Students must demonstrate understanding of the concept of a function and perform the algorithm for finding the inverse.

ASSESSMENT SUGGESTION

Knowledge and Understanding

Mark on a scale of 1, 2 or 3.

1: Student does not make a reasonable attempt and does not show understanding.

2: Student makes a reasonable effort, showing some understanding.

3: Student's work is exemplary.

Return the responses with comments to students to help improve their future performance, where necessary.

Level 3 Performance

Responses should include a correct expression for the inverse, appropriate function notation, and an indication that students understand the effect of restricting the domain.

15. Communication, p. 276

Management Suggestion

Invite partners to trade and proofread each other's responses. Exemplary responses might be nominated for reading aloud to the class.

Achievement Chart Connection

Students should present a clear, justified response, using mathematical symbols and conventions.

Level 3 Performance

Students should demonstrate understanding of the concept that each point of the original function corresponds to one point of the inverse. Responses should show symbols and conventions such as $f(x)$ and $f^{-1}(x)$, used accurately.

17. Application, p. 276

Management Suggestion

Invite students to work in pairs. Those who finish quickly might enjoy giving, or presenting, a coded number to another pair of students to decode.

Achievement Chart Connection

Students apply their knowledge of codes, functions, and inverses in unfamiliar situations.

Level 3 Performance

Students should develop accurate equations for coding and decoding. They should also understand and demonstrate the effect of restrictions.

18. Check Your Understanding, p. 276

Management Suggestion

Use a graphing program or the TI-83 Plus with a view screen to project the image of a graph onto the board. Enter a function and project the graph. Invite a volunteer to go to the board and use chalk to draw the inverse. After discussion, enter a new function and repeat.

19. Thinking, Inquiry, Problem Solving, p. 277

Management Suggestion

Allow students to work in small groups to prepare a response to the problem. Presentations might be posted on the bulletin board or shared using overhead acetates prepared by each group.

Achievement Chart Connection

Students must respond to a complex question with many components.

Level 3 Performance

Students should find a reasonable equation for the profit in terms of quantity and for the inverse. They should also develop a reasonable strategy and provide relevant calculations.

ASSESSMENT SUGGESTION

Teamwork, Initiative

During small group work, observe students' efforts and interaction and collect data for assessment.

The Chapter Problem—Cryptography

The problem is a natural environment for restrictions on the domain. Encourage the students to experiment with input values greater than 26 (perhaps 27, 28, 29, etc.), and compare the output with the output produced for all values of the domain. Encourage students to sketch the graph of the encoder and its inverse in CP14 to help them understand the problem with the code.

The Chapter Problem Answers are on p. 3-45 of the *Teacher Resource*.