

4.2 Maximum and Minimum Values of Quadratic Functions

Specific Expectations

The following expectation is common to both the Functions and Relations and Functions courses.

- OC1.03 determine the maximum or minimum value of a quadratic function whose equation is given in the form $y = ax^2 + bx + c$, using the algebraic method of completing the square

Evidence of Learning

Students should be able to

- change the equation of a quadratic function from standard form to vertex form by completing the square
- use translations, x -intercepts, and symmetry to locate the vertex of a parabola
- interpret an equation in factored form to determine x -intercepts
- read the vertex from the vertex form of a quadratic equation and relate it to the maximum or minimum value of the function
- write an equation in different forms to get specific information from each form

Text Page 308

Time Required Functions and Relations:

75 min

Time Required Functions: 150 min

Materials and Equipment

- graphing calculator

Overview

- Use a quadratic function to model a business example involving profit, cost, and revenue.
- Manipulate equations of quadratic functions.
- Transform a function instead of completing the square in some cases to find the vertex.

Management Options

Groups – Form students in groups of three or four to work through the *Think, Do, Discuss* questions in Part 1, provided that there is at least one graphing calculator per group. You might prepare posters outlining the steps for completing the square or prepare “checkpoints.” For the checkpoints in Part 1, list the functions for $p(x)$, $R(x)$, $C(x)$, and $P(x)$ on separate cards. Each team leader can monitor the progress of the group by referring to the checkpoints.

At the end of the section, present an equation. Put the following cards in an envelope and ask each group member to draw one card. Instruct students to find the form or item and compare results.

completed square vertex form
intercept(s) factored form
standard form sketch

Teacher Demonstration – Using an appropriate example, present the various forms for the equation of a parabola. Emphasize with colour the information that is visible in each form. Present a question on the overhead or on the board and develop equations in standard form, vertex form, and factored form. Have students graph each form. Take time to discuss the pros and cons of each form in the context of the question.

EXTENSION

Ask students to consider the following question.

1. Create an original situation, which can be modelled using a quadratic equation, and trade with a partner. Allow your partner to analyze the information you provide and then discuss the results obtained and the results you anticipated.

Part 1: Changing a Quadratic Function from Standard Form to Vertex Form by Completing the Square

Think, Do, Discuss, p. 308

- The selling price of an item depends, in part, on how many items the company expects to sell. The relation between the price of an item, p , and the number of items sold, x , is called the **demand function**, $p(x)$. Through research, RAYZ Manufacturing collected the following data, where x is the number of pairs of sunglasses sold in thousands and p is the price of one pair in dollars.

Number of Sunglasses Sold, x (thousands)	0.7	1.4	1.8
Price, p (\$)	30.00	27.50	25.00
Number of Sunglasses Sold, x (thousands)	2.1	2.9	3.2
Price, p (\$)	22.50	20.00	17.50

Describe the relation between the price and the number of items sold. How does the price change as the number of items sold increases? Does this change make sense to you? Explain your answer.

As the number of items sold increases, the price decreases. The rate of decrease is not constant but lessens. This change makes sense, because a portion of the price is the cost of overhead and, as the number of items increases, this cost decreases because it is spread over more items. But the overhead cost will become constant at a certain point, no matter how many items are sold.

- Enter the data into a graphing calculator. Then perform a linear regression to obtain the demand function, $p(x)$. Round the coefficients to whole numbers.

LinReg
$y=ax+b$
$a=-4.973169797$
$b=33.77922576$
$r^2=.9832667141$
$r=-.9915980608$

$$p(x) = -5x + 34$$

- Write the **revenue function**, $R(x)$, based on your demand function. Recall that the revenue is the income from sales.

$$\begin{aligned} R(x) &= \text{number of pairs of glasses sold} \\ &\quad \times \text{price of each item} \\ &= x \times \text{demand function} \\ &= x \times p(x) \\ &= ? \end{aligned}$$

$$\begin{aligned} R(x) &= x \times p(x) \\ &= x(-5x + 34) \\ &= -5x^2 + 34x \end{aligned}$$

- Research for this item shows that the **cost function** is $C(x) = 3x + 35$. Obtain the **profit function** using $P(x) = R(x) - C(x)$. Simplify the right side by expanding and then collecting like terms. What kind of function is the profit function? Describe the graph of $y = P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-5x^2 + 34x) - (3x + 35) \\ &= -5x^2 + 31x - 35 \end{aligned}$$

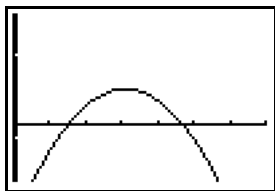
This function is quadratic. The graph is a parabola opening down.

- Complete the square of the profit function to determine how many pairs of glasses should be sold to maximize profits.

$$\begin{aligned} P(x) &= -5x^2 + 31x - 35 \\ &= -5\left(x^2 - \frac{31}{5}x\right) - 35 \\ &= -5\left(x^2 - \frac{31}{5}x + \frac{961}{100} - \frac{961}{100}\right) - 35 \\ &= -5\left(x^2 - \frac{31}{5}x + \frac{961}{100}\right) + \frac{961}{20} - 35 \\ &= -5\left(x - \frac{31}{10}\right)^2 + \frac{261}{20} \end{aligned}$$

The maximum value of the function can be determined from the vertex, $\left(\frac{31}{10}, \frac{261}{20}\right)$. When $x = 3.1$, that is, when 3100 items are sold, profit will be maximized.

6. Sketch the graph of the profit function. What are the coordinates of the y-intercept? Which form of the equation, standard form or vertex form, did you use to find the y-intercept? Explain the significance of the x-intercepts. For the graph of a profit function, the x-intercepts are called the **break-even points**. Explain this description.



The coordinates of the y-intercept are $(0, -35)$. Find the y-intercept by substituting $x = 0$ in the standard form of the equation. When $x = 0$, $y = -35$.

For each x-intercept, the corresponding y-value is 0, that is, the profit is 0.

A negative value for profit means that costs are greater than revenue and the company loses money. So profit should be positive, so the company will make money. When profit is 0, revenue equals cost and the company breaks even.

7. Use the profit function, along with the quadratic formula, to find the break-even quantities. Remember that x is in thousands.

For $P(x) = -5x^2 + 31x - 35$, $a = -5$, $b = 31$, and $c = -35$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-31 \pm \sqrt{31^2 - 4(-5)(-35)}}{2(-5)}$$

$$x \approx 1.5 \quad \text{and} \quad x \approx 4.7$$

The break-even quantities are about 1500 items and about 4700 items. If the company sells more than 1500 items or less than 4700 items, it will make a profit, because the graph of the function is above the horizontal axis.

8. Use a graphing calculator to check your graph. Check the vertex coordinates and the break-even points using the maximum and zero operations in the **CALC** menu.

TECHNOLOGY SUGGESTION

Encourage some students to find the maximum and zeros of the function by experimenting with the TRACE function on the graphing calculator.

Part 2: Finding the Vertex without Completing the Square

Think, Do, Discuss, p. 309

1. In Chapter 3, you did some work on transforming functions. If you apply a vertical translation to a parabola, how do the coordinates of the vertex change? Consider the parabola shown here. Its equation is $y = a(x + 2)(x - 5)$. What is the x-coordinate of the vertex? How do you know what the x-coordinate is?

The x-value of the vertex is unchanged, but the y-value changes, depending on the size of the vertical translation. The x-coordinate of the given parabola is 1.5, because it must be on the axis of symmetry, which is halfway between the zeros of -2 and 5 .

2. Use the coordinates of the y-intercept to find the value of a . Then expand the equation of the parabola so that it is in the form $y = ax^2 + bx + c$.

When $x = 0$, $y = -10$, so

$$-10 = a(0 + 2)(0 - 5)$$

$$a = 1$$

Then

$$y = 1(x + 2)(x - 5)$$

$$y = x^2 - 3x - 10$$

3. What vertical translation would make the parabola pass through the origin? What is the equation of the translated parabola?

Translate the parabola vertically up 10 units so that the parabola will pass through the origin.

The equation of the translated parabola is

$$y - 10 = x^2 - 3x - 10$$

or

$$y = f(x) + 10$$

$$y = (x^2 - 3x - 10) + 10$$

$$y = x^2 - 3x$$

4. Express the new equation in factored form. What are the new x -intercepts? Why is the x -coordinate of the vertex for the new parabola the same as that for the original parabola? Use these new x -intercepts to explain your answer.

$y = x(x - 3)$, so the new x -intercepts are 0 and 3. The new parabola has been shifted vertically but not horizontally, so its vertex has the same x -value. The x -value is still on the axis of symmetry, halfway between the x -intercepts, so it is 1.5.

5. Now consider the graph of the profit function, $P(x) = -5x^2 + 31x - 35$, from Part 1. What vertical translation would make the graph pass through the origin? What is the equation of the translated parabola?

The y -intercept is -35 , so a vertical translation of 35 units up would cause the graph to pass through the origin. The equation of the translated parabola is $y = -5x^2 + 31x$.

6. Express the new equation in factored form. Where are the new x -intercepts? What is the x -coordinate of the vertex? Use the original equation to find the y -coordinate of the vertex for the graph of the profit function. Compare your answer to your result in Part 1.

$$y = -5x\left(x - \frac{31}{5}\right)$$

The new x -intercepts are 0 and $\frac{31}{5}$. The x -coordinate of the vertex is halfway between these values and is $\frac{31}{10}$. The x -coordinate of the vertex does not change with a vertical

translation, so substitute $x = \frac{31}{10}$ in the equation for the profit function and solve for y .

$$y = -5x^2 + 31x - 35$$

$$y = -5\left(\frac{31}{10}\right)^2 + 31\left(\frac{31}{10}\right) - 35$$

$$y = 13.05$$

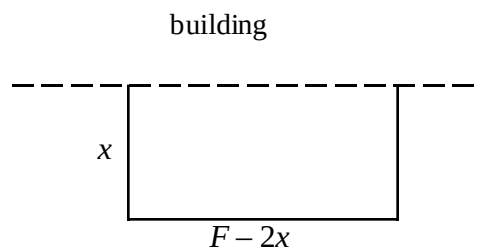
The answer, 13.05, is equal to the result in Part 1, $\frac{261}{20}$.

Additional Example: Functions and Relations

A rectangular area beside a building is to be fenced with a specific amount of fencing. Find the maximum area that can be fenced and the dimensions of that area.

Solution

Let F be the length of available fencing.



Area is a function of x .

$$\begin{aligned} A(x) &= (x)(F - 2x) \\ &= -2x^2 + Fx \\ &= -2x\left(x - \frac{F}{2}\right) \end{aligned}$$

The x -intercepts are 0 and $\frac{F}{2}$, so the x -coordinate of the vertex is $\frac{F}{4}$.

For $x = \frac{F}{4}$,

$$\begin{aligned}
 A(x) &= -2x^2 + Fx \\
 &= -2\left(\frac{F}{4}\right)^2 + F\left(\frac{F}{4}\right) \\
 &= -2\left(\frac{F^2}{16}\right) + \frac{F^2}{4} \\
 &= \frac{F^2}{8}
 \end{aligned}$$

The maximum area will be one-eighth of the length of available fencing squared. The width of the area will be one-quarter of the length of available fencing and the length will be one-half of the available fencing.

Additional Example: Functions

A resort hotel has 250 rooms and the standard rate is \$150 per night per room. During the non-holiday season, 100 rooms are rented each day on average, but, for each \$20 price reduction, 25 more rooms will be rented. What non-holiday price should be advertised?

Solution

Let x represent the number of price changes.

$$\begin{aligned}
 \text{income} &= (\text{number rooms rented}) \\
 &\quad \times (\text{price per room}) \\
 &= (100 + 25x)(150 - 20x) \\
 &= -500x^2 + 1750x + 15\,000 \\
 &= -500(x - 1.75)^2 + 16\,531.25
 \end{aligned}$$

For maximum income, there are 1.75 price changes, so the advertised price should be

$$\$150 - (1.75)(\$20) = \$115$$

ASSESSMENT SUGGESTION

Knowledge and Understanding

Provide a quadratic function and invite students to manipulate it to provide as much information as possible. Repeat this activity each day for the next few days.

ASSESSMENT SUGGESTION

Communication

Invite students to prepare a flow chart or mind map to show all the forms of a quadratic equation and the information in each form. Students should show the connections between each equation and a graph.

Practise, Apply, Solve 4.2

Questions with a Functions and Relations Focus:

2–6, 8–21

Questions with a Functions Focus:

1–7, 11–18, 20

6. Knowledge and Understanding, p. 314

Management Suggestion

Set aside class time for individual work on this problem. Collect and assess responses.

Achievement Chart Connection

This straightforward question allows students to demonstrate understanding of the concepts.

Level 3 Performance

Students must state accurate and appropriate functions and correctly complete the square. Graphs should match the algebra. Students' interpretation of the algebraic results must be valid.

11. Communication, p. 314

Management Suggestion

Allow students to trade work with a partner for peer review. Expect students to make constructive suggestions to clarify their explanations.

Achievement Chart Connection

Students must combine mathematical language and symbols to explain their thinking.

Level 3 Performance

Students should clearly refer to two functions using appropriate notation and state and support a hypothesis.

17. Thinking, Inquiry, Problem Solving, p. 316

Management Suggestion

Provide large chart paper and have pairs or triplets of students work together to create a display. Set a time, for example, 15 min, for this task.

Achievement Chart Connection

Encourage students to apply these steps of the problem-solving process – forming a hypothesis, collecting data, analyzing and drawing a conclusion, and communicating results.

Level 3 Performance

Students must develop and organize data and reach a reasonable conclusion based on the available information.

18. Check Your Understanding, p. 316**Management Suggestion**

Work with the class to prepare a large overview on the bulletin board — a roll of picnic-table paper is a useful covering for the board. Then give students time to record a succinct version in their own words in their notebooks.

20. Application, p. 317**Management Suggestion**

Discuss this question as a whole class before individual work begins to ensure that all students have ideas about how to begin.

Achievement Chart Connection

Students are asked to apply the concept of a maximum in an unfamiliar and complex situation.

Level 3 Performance

Students should create a reasonable expression for area in terms of the dimensions and an algebraic or graphic method for finding the maximum area.

The Chapter Problem Answers are on p. 4-63 of the *Teacher Resource*.

The Challenge Answers are on p. of the *Teacher Resource*.