

4.4 Introducing Complex Numbers

Specific Expectations

The following expectations are common to both the Functions and Relations and Functions courses.

- OC1.04 identify the structure of the complex number system and express complex numbers in the form $a + bi$, where $i^2 = -1$ (e.g., $4i$, $3 - 2i$)
- OC1.05 determine the real or complex roots of quadratic equations, using an appropriate method (e.g., factoring, the quadratic formula, completing the square), and relate the roots to the x -intercepts of the graph of the corresponding function

Evidence of Learning

Students should be able to

- * distinguish the characteristics of number sets
- * express imaginary numbers in $a + bi$ form
- * find complex roots of quadratic equations by using the quadratic formula
- * interpret the presence of complex roots graphically as the absence of x -intercepts

Text Page 328

Time Required Functions and Relations or Functions: 75 min

Materials and Equipment

- graphing calculator

Overview

- Learn about complex numbers.
- Solutions of quadratic equations include complex numbers, expressed in $a + bi$ form.

MANAGEMENT SUGGESTION

Use overlapping coloured shapes of various sizes to model the increasing breadth of number sets.

Management Options

Groups – After initial, brief whole-class discussion, pairs or triplets of students could spend a few minutes answering the *Think, Do, Discuss* questions in Part 1.

Teacher Demonstration – It is a good idea when students first encounter complex numbers to provide an explicit definition and examples. Work through the *Think, Do, Discuss* questions of Part 2, providing examples of complex numbers.

EXTENSION

Ask students to research to find how complex numbers are graphed.

Part 1: Familiar Number Systems

Think, Do, Discuss, p. 328

1. *Lia has some CDs and buys four more. She now has 28 CDs. How many CDs did she have originally? Write an equation. Let the variable be the original number of CDs. What is the solution? What kind of numbers did you use in your solution?*

$$x + 4 = 28$$

$$x = 24$$

Lia originally had 24 CDs. The solution is a natural number.

2. *Craig builds seven lawn chairs in two days. How many chairs does he build in a day? Write an equation. Let the variable be the original number of chairs that Craig builds in a day. Solve your equation. What kind of numbers did you use in your solution?*

$$2x = 7$$

$$x = 3.5$$

Craig builds 3.5 chairs in a day. In this case, the solution is a rational number, but must be greater than or equal to 0.

3. *Stuart buys nine CDs. He then has a total of nine CDs. How many CDs did he have originally? Write an equation. Let the variable be the original number of CDs. What is the solution? This number was not introduced into use until the tenth century!*

$$\begin{aligned}9 + x &= 9 \\ x &= 0\end{aligned}$$

Stuart did not have any CDs originally.

4. *The temperature rose 10°C and is now 6°C . What was the initial temperature? Write an equation. Let the initial temperature be the variable. Solve the equation. What kind of numbers did you use in your solution?*

$$\begin{aligned}x + 10 &= 6 \\ x &= -4\end{aligned}$$

The initial temperature was -4°C .

The solution is a negative number.

5. *Kim wants to create a square enclosure for her vegetable garden with an area of 2 m^2 . How much chicken wire will she need for fencing the garden? Write an equation. Let the variable be the length of each side of the square enclosure. Solve the equation. What type of number did you use in your solution?*

$$\begin{aligned}x \times x &= 2 \\ x^2 &= 2 \\ x &= \pm\sqrt{2}\end{aligned}$$

Kim will need about 1.4 m of chicken wire.

The solution is an irrational number.

6. *The above questions refer to the following sets of numbers:*

the set of natural numbers, N , $\{1, 2, 3, 4, \dots\}$

the set of whole numbers, W , $\{0, 1, 2, 3, 4, \dots\}$

the set of integers, I , $\{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$

the set of rational numbers, Q ,

$$\left\{\frac{a}{b} : a, b \in I, b \neq 0\right\}$$

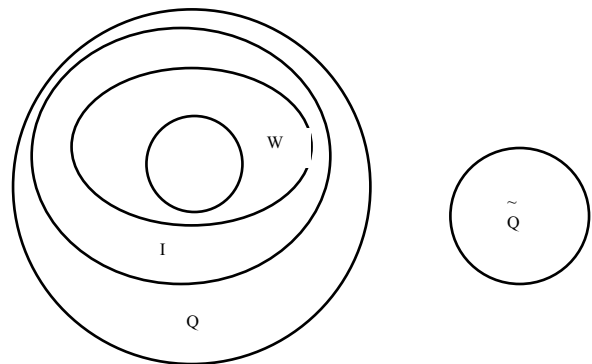
the set of irrational numbers, $\tilde{Q}$. An irrational

number cannot be expressed as the ratio of two integers, for example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\dots$.

Are these sets distinct, or do some numbers belong to more than one set? Give examples to illustrate your answer. Draw a diagram to show how the sets connect with each other.

Some numbers belong to several sets, for example, 5 is a whole number and also an integer.

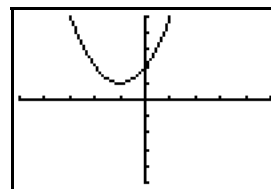
<add N, cap & bold, to middle circle)



Part 2: Defining the Set of Complex Numbers

Think, Do, Discuss, p. 329

1. *Sketch the graph of the function $f(x) = x^2 + 2x + 2$. How many zeros does $f(x)$ have?*



This graph has no zeros.

2. Try to find the zeros of the function $f(x) = x^2 + 2x + 2$ by letting $f(x) = 0$ and using the quadratic formula.

For $x^2 + 2x + 2$, $a = 1$, $b = 2$, and $c = 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{-4}}{2}$$

3. Why are you unable to find the zeros?

Cannot solve the square root of a negative number.

4. Let a non-real number be i , such that $i^2 = -1$. Then you can write any negative number as a multiple of i^2 . For example, you could write -4 as $4i^2$ and write -31 as $31i^2$, and so on. Now you can solve the equation $x^2 + 4 = 0$.

$$x^2 + 4 = 0$$

$$x^2 = -4$$

$$x^2 = 4i^2$$

$$x = \pm 2i$$

What are the solutions of $x^2 + 25 = 0$? of $x^2 + 3 = 0$?

$$x = \pm 5i$$

$$x = \pm \sqrt{3}i$$

5. The number i has been called an **imaginary number**. This name is easy to remember, but it is misleading because the number i does exist. Some quadratic equations have solutions that have real and imaginary parts. Try to solve the equation $x^2 + 2x + 2 = 0$. Use the number i to help you obtain two roots.

$$x = \frac{-2 \pm \sqrt{-4}}{2}$$

$$x = \frac{-2 \pm 2i}{2}$$

$$x = -1 + i \quad \text{and} \quad x = -1 - i$$

6. The roots in step 5 should be in the form $a + bi$ and $a - bi$. These numbers are called **complex numbers**. Notice that complex numbers have a real part, a , and an imaginary part, $\pm bi$. Would you describe real numbers as complex? Explain your answer.

Any real number could be expressed as a complex number such that the coefficient of the imaginary part is 0. For example, express 5 as the complex number $5 + 0i$.

ASSESSMENT SUGGESTION

Communication

Ask students to select a good example of their work that shows the solution of an equation with complex roots and the corresponding graph. Collect the examples and assess them.

Additional Example: Functions and Relations

Predict the roots of $y = (x + 3)(3x^2 + x + 5)$. Graph the equation and describe the solution in terms of the graph.

Solution

$$(x + 3)(3x^2 + x + 5) = 0$$

Set each factor to 0.

$$x + 3 = 0$$

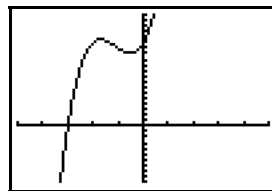
$$x = -3$$

For $3x^2 + x + 5 = 0$, $a = 3$, $b = 1$, and $c = 5$.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(3)(5)}}{2(3)}$$

$$x = \frac{-1 + i\sqrt{59}}{6} \quad \text{and} \quad x = \frac{-1 - i\sqrt{59}}{6}$$

The graph crosses the x -axis at only one real point, $(-3, 0)$.



Additional Example: Functions

Find the solutions for each equation and comment on the differences caused by one change!

$$x^2 - 5x - 24 = 0$$

$$x^2 - 5x + 24 = 0$$

Solution

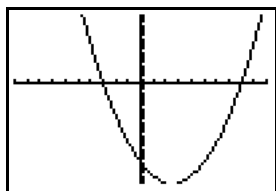
Factor the first equation.

$$x^2 - 5x - 24 = 0$$

$$(x - 8)(x + 3) = 0$$

The solutions are $x = 8$ and $x = 3$.

Using the quadratic formula would give the same results.

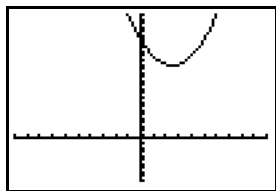


The second equation cannot be factored, so use the quadratic formula to find the roots.

For $x^2 - 5x + 24 = 0$, $a = 1$, $b = -5$, and $c = 24$.

$$\begin{aligned} x &= \frac{5 \pm \sqrt{25 - 4(1)(24)}}{2(1)} \\ &= \frac{5 \pm \sqrt{-71}}{2} \end{aligned}$$

Because both roots are complex, the graph of this function does not have x -intercepts.



ASSESSMENT SUGGESTION

Works Independently

Solving equations should be familiar to students, so students should adapt quickly to working with complex numbers. Help students who need assistance, and note when students become independent.

Practise, Apply, Solve 4.4

Questions with a Functions and Relations Focus:

1–5, 9–18, 20, 21

Questions with a Functions Focus:

1–6, 9–12, 15–18

9. Knowledge and Understanding, p. 332

Management Suggestion

Encourage students to use different strategies for finding the roots of a quadratic equation, rather than relying on just one strategy.

Achievement Chart Connection

Students should factor, use the quadratic formula, or use algebra to find roots.

Level 3 Performance

To find roots, students should use the simplest appropriate method, for example, factoring. Complex roots should be expressed in the form $a + bi$.

15. Application, p. 333

Management Suggestion

Give individuals time to consider the problem and attempt a solution. Then invite partners to discuss solutions and ask for volunteers to present a solution, discussing each step.

Achievement Chart Connection

Students must apply the concepts in this section to a familiar situation.

Level 3 Performance

Students must create a reasonable graph of the function, showing that the solution is $n(t) > 65\,000$. Students should support their justifications.

16. Communication, p. 333

Management Suggestion

Be aware that students favour different communication modes — some will want to write responses while others will feel more comfortable presenting oral answers.

Achievement Chart Connection

Students should use mathematical symbols and conventions and their own words to discuss complex numbers.

Level 3 Performance

Students refer to the $a + bi$ form and connect complex numbers to real numbers.

17. Thinking, Inquiry, Problem Solving, p. 333

Management Suggestion

Assign the question as homework and discuss answers at the beginning of the next class. For students who are planning on post-secondary study, point out the value of looking ahead by identifying future topics.

Achievement Chart Connection

Students investigate operations with complex numbers, as indirect preparation for subsequent lessons. They should follow these problem-solving steps: hypothesize, collect data, draw conclusions, and generalize.

Level 3 Performance

Students should create several examples of a sum and a product, state a hypothesis, and base their conclusions on the data they have generated.

18. Check Your Understanding, p. 333

Management Suggestion

Provide a few minutes near the end of class for students to prepare and share responses. Students should record the combined “important ideas.”

The Chapter Problem—Financial Planning

The Chapter Problem Answers are on p. 4-63 of the *Teacher Resource*.

Challenge 1

Challenge 2

The Challenge Answers are on p. 4-68 of the *Teacher Resource*.