

4.6 Reciprocal Functions

Specific Expectations

The following expectations are common to both the Functions and Relations and Functions courses.

- OC2.03 determine, through investigation, the properties of the functions defined by $f(x) = \sqrt{x}$ [e.g., domain, range, relationship to $f(x) = x^2$] and $f(x) = \frac{1}{x}$ [e.g., domain, range, relationship to $f(x) = x$.]
- OC2.04 explain the relationship between a function and its inverse (i.e., symmetry of their graphs in the line $y = x$; the interchange of x and y in the equation of the function; the interchanges of the domain and range), using examples drawn from linear and quadratic functions, and from the functions $f(x) = \sqrt{x}$ and $f(x) = \frac{1}{x}$
- OC2.05 represent inverse functions, using function notation, where appropriate

Evidence of Learning

Students should be able to

- write an algebraic expression for a reciprocal function
- graph a function and its reciprocal function
- identify examples of inverse variation
- identify asymptotes and describe the graph of the function near an asymptote

Text Page 338

Time Required Functions and Relations or Functions: 75 min

Materials and Equipment

- graphing calculator

Overview

- Introduce the reciprocal function $f(x) = \frac{1}{x}$.
- Connect restrictions on variables to asymptotes in the graph.
- A discussion of domain, range, intersecting curves, and the transformation of functions provides a rich exploration and permits students to consolidate learning from previous work in preparation for a calculus course in grade 12.

MANAGEMENT SUGGESTION

If possible, find examples of reciprocal functions in other courses, for example, chemistry, physics, or biology. Discuss the corresponding graph and algebra for each example. This section prepares students for working with hyperbolas in Chapter 7U.

Management Options

Groups – Arrange with the science teachers to share common material or spend part of a class collecting data showing inverse variation. When students recognize the importance of a constant product in various applications, the value of studying this type of function will be evident. Students can work in groups of three on the *Think, Do, Discuss* questions for each part. Arrange to have “emissaries” move to new groups at predetermined intervals to share learning. Finish the class with a whole-class discussion or a short assignment to be collected and assessed.

Teacher Demonstration – Begin the class with a problem derived from a current situation in your school. For example, for fundraising, increasing the number of donors while decreasing the donation yields a constant amount. Invite students to describe other examples of inverse variation. Create algebraic descriptions of the functions and develop graphs for these examples, drawing attention to the connections between the algebra and the graphs.

Independent Work – Allow students to work through the *Think, Do, Discuss* question of each part. Then form students into groups of four in which they can compare responses and discuss questions. To ensure that students are involved in varied groups, make a class set of cards so that each card has a different symbol — a number, letter, shape, or colour. Distribute one card to each student. Using any criteria, sort students into random groups.

EXTENSION

Use *Think, Do, Discuss* questions 4 to 10 of Part 3 for extension.

Part 1: Comparing Hourly Rates

Think, Do, Discuss, text page 338

1. What information does Stuart need to calculate the hourly rate for painting houses?

Stuart needs to know the average number of hours required to paint a house.

2. If Stuart takes 40 h to paint a house, what is the hourly rate?

The hourly rate is $\$450 \div 40 = \$11.25$.

3. Suppose Stuart wants to determine the hourly rate of pay if he takes less than 40 h to paint a house, or if he takes more than 40 h to paint a house. Copy and complete the table to show the hourly rate, y , for the hours worked, x , to paint one house.

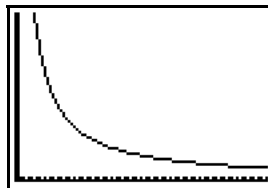
Hours Worked, x (h)	Hourly Rate, y (\$/h)
5	90.00
10	45.00
15	30.00
20	22.50
25	18.00
30	15.00
35	12.86
40	11.25
45	10.00
50	9.00
55	8.18

4. What happens to the hourly rate as the number of hours increases? Explain why the hourly rate is represented by y and why the number of hours worked is represented by x .

As the number of hours increases, the hourly rate decreases, because the total amount remains constant. The hourly rate depends on the number of hours worked.

5. Write an equation for the relationship between the hourly rate, y , and the number of hours worked, x . Enter the equation into a graphing calculator and graph it. Restrict the domain to only positive x -values.

$$y = \frac{450}{x}$$



6. Describe the graph. According to the graph, what happens to y as x becomes very small, that is, close to 0? What happens to y as x becomes very large?

As x becomes very small, approaching 0, the value of y increases very quickly. As x becomes very large, the value of y becomes very small, remaining positive and approaching 0. Any point on the graph has two positive coordinates.

7. How would the graph change if the rate of pay increased to \$550 per house?

The new graph would be similar, but the y -values for this graph are slightly larger than the corresponding y -values of the first graph. If the hours required do not change but the total cost for a specific job increases, the hourly rate will increase.

8. For each line in the table, calculate the product xy . What do you notice? Why does this happen? This type of relationship is called **inverse variation**. What other quantities vary in this way?

The product is always 450. The total cost for the job is constant, so, as the number of hours increases, the hourly rate decreases.

Other quantities vary inversely such as the following:

- In a sealed container, internal pressure increases as volume decreases.
- As the number of students in a class increases, the interaction time between the teacher and individual students decreases.

9. What factors, other than the speed at which Stuart can paint, might influence the time it takes to paint a house? Which job would you advise Stuart to take? Explain your answer.

The amount of preparation required (scraping off old or damaged paint, primer coats) the amount of detailed work (windows, trim colours), and weather conditions are some factors that would affect the speed at which Stuart can paint.

Students should justify their advice; for example, Stuart should choose the shipping job because that job will not be affected by weather conditions. If Stuart could paint a house in about 39 h, he would earn an equivalent hourly rate, but it would take more than 39 h to paint the average house.

Part 2: The Reciprocal

Function $f(x) = \frac{1}{x}$

Think, Do, Discuss, text page 339

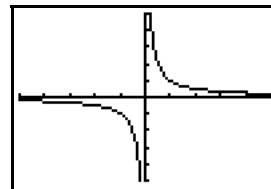
1. $f(x) = \frac{1}{x}$ is a **reciprocal function**.

Explain why this name is appropriate.

When the numerator and denominator of a fraction are interchanged, the result is a reciprocal. To reciprocate means to interchange. For this function, the y -value is the reciprocal of the x -value.

2. Enter the function into a graphing calculator and adjust the window ($Xmin=-5$, $Xmax=5$, $Ymin=-5$, and $Ymax=5$). Then graph. Describe the shape of the graph and any special features. This graph is called a **hyperbola**.

The graph has two parts — the parts are mirror images. The arms of the graphs get very close to the axes, but they do not appear to cross the axes. Some students may recognize the graph is symmetric about the line $y = x$.



3. What happens to the value of y as x becomes very large, for example, greater than 1000? What happens to y as x becomes very small, for example, less than -1000 ? Describe the graph for both situations. Does the graph meet or cross the x -axis? How do you know?

In both cases, the value of y gets very close to 0. The graph does not meet or cross the x -axis, because the value of the fraction $\frac{1}{x}$ could never be 0. No matter how large the value of x becomes, the value of this fraction will

always be positive. Similarly, no matter how small x becomes, the value of $\frac{1}{x}$ will always be negative. So the graph will approach but never meet or cross the x -axis.

4. What is true about the value of y when

(a) x is close to 1?

y is close to 1.

(b) x is close to -1 ?

y is close to -1 .

5. What happens to the value of y as x decreases from 1 to 0? as x increases from -1 to 0?

As x decreases from 1 to 0, y increases dramatically, becoming quickly larger. As x increases from -1 to 0, y decreases dramatically, becoming quickly larger. The sign in each case does not change.

6. In the table, what is the y -value for $x = 0$? What does this y -value mean? What is the domain for the function $f(x) = \frac{1}{x}$? What is the range?

For $x = 0$, y is $\frac{1}{0}$, and division by 0 is undefined. The domain is $\{x \mid x \neq 0, x \in \mathbf{R}\}$. The range is $\{y \mid y \neq 0, y \in \mathbf{R}\}$.

7. As x gets closer and closer to 0, the graph approaches the y -axis but never meets it. The y -axis is an example of a **vertical asymptote**. The graph also has a **horizontal asymptote**. What is its equation? How do the equations of the asymptotes relate to the domain and the range of f ? Since the asymptotes are perpendicular for this hyperbola, it is called a **rectangular hyperbola**.

The horizontal asymptote has equation $y = 0$. The asymptotes match the restrictions on the domain and range of the function.

8. Find the equation for the inverse of $f(x) = \frac{1}{x}$, that is, $f^{-1}(x)$. What do you notice? Does this result seem reasonable from the graph? Explain your answer.

When x and y are interchanged, the inverse is identical to the original function. This result is consistent with the graph, because an inverse is a reflection about the line $y = x$ and the original function itself was symmetrical about this line.

9. The **inverse** of a function is written as $f^{-1}(x)$. However, $[f(x)]^{-1}$ refers to the **reciprocal function**. Explain the difference between the inverse of a function, $f^{-1}(x)$, and the reciprocal of a function, $\frac{1}{f(x)}$. Use examples of each function and discuss the domain and the range in each case.

To create the inverse, interchange the x and y . But, to create a reciprocal, interchange the numerator and denominator of the original function.

If $f(x) = x + 2$, the inverse function is $f^{-1}(x) = x - 2$.

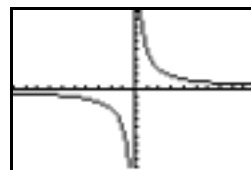
The domain of both functions is $\{x \mid x \in \mathbf{R}\}$. The range of both functions is $\{y \mid y \in \mathbf{R}\}$.

The reciprocal function is $[f(x)]^{-1} = \frac{1}{x + 2}$.

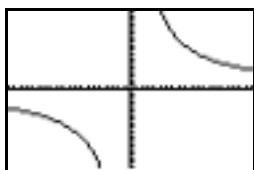
The domain is $\{x \mid x \neq -2, x \in \mathbf{R}\}$. The range is $\{y \mid y \neq 0, y \in \mathbf{R}\}$.

10. Using your knowledge of transformations, how would the graph of each function compare to the graph of $f(x) = \frac{1}{x}$? Check using a graphing calculator.

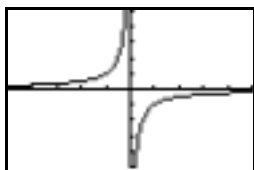
(a) $f(x) = \frac{5}{x}$ [Hint: $\frac{5}{x} = 5\left(\frac{1}{x}\right)$]



(b) $f(x) = \frac{100}{x}$



(c) $f(x) = \frac{-1}{x}$



11. Write each function in question 10 as a power of x . In this form they are power functions.

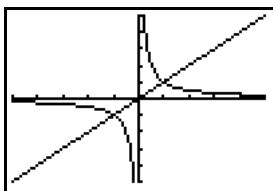
(a) $f(x) = 5x^{-1}$ (b) $f(x) = 100x^{-1}$

(c) $f(x) = -x^{-1}$

Part 3: Making Connections Between the Graphs of Functions and Their Reciprocals

Think, Do, Discuss, text page 340

- On a graphing calculator, adjust the window to ($Xmin=-5$, $Xmax=5$, $Ymin=-5$, and $Ymax=5$). Graph $y = x$ and $y = \frac{1}{x}$ on the same axes. At what points do the graphs intersect? What are the y -coordinates of the points of intersection? Explain why these values make sense.



The graphs intersect at $(1, 1)$ and $(-1, -1)$. These values make sense, because the value of the reciprocal function approaches the value of independent variable (x) at these points. See *Think, Do, Discuss* question 4 in Part 2.

- Where is the zero for the function $y = x$?

What feature of the graph of $y = \frac{1}{f(x)}$

exists at this x -value? Explain why this makes sense.

The zero is at the origin. For the reciprocal function, an asymptote occurs at this zero, because division by 0 is undefined.

- What do you notice about the quadrants in which each graph occurs? When $y = x$ is close to the x -axis, where is $y = \frac{1}{f(x)}$? When $y = x$ is far away

from the x -axis, where is $y = \frac{1}{f(x)}$?

Both graphs occur in quadrants I and III.

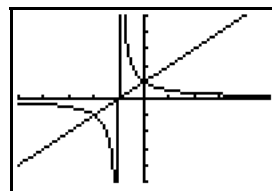
When $y = x$ is close to the x -axis, the reciprocal function is far away from the x -axis. When $y = x$ is far away from the x -axis, the y -value of the reciprocal function is close to the x -axis.

- Repeat steps 1 to 3 for each pair of functions. When you enter the reciprocal functions into a graphing calculator, you will need to include brackets around each denominator. For example, enter the function

$$y = \frac{1}{x+1} \text{ as } Y2=1/(X+1).$$

Students can find the intersection points by inspection, using TRACE, or by interpreting transformations.

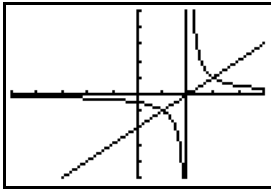
(a) $y = x + 1$ and $y = \frac{1}{x+1}$



The graphs intersect at $(0, 1)$ and $(-2, -1)$. The original and reciprocal functions have been translated 1 unit left. The zero of the original function

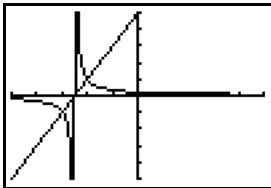
occurs at $(-1, 0)$, so the vertical asymptote is $x = -1$. When the original function is close to the x -axis, the reciprocal is far away from this axis. When the original function is far away from the x -axis, the reciprocal is close to this axis.

(b) $y = x - 2$ and $y = \frac{1}{x-2}$



The graphs intersect at $(3, 1)$ and $(1, -1)$. These functions have been translated 2 units right, compared to the functions $y = x$ and $y = \frac{1}{x}$. The zero for $y = x - 2$ occurs at $(2, 0)$ and the asymptote is $x = 2$. When the original function is close to the x -axis, the reciprocal is far away and vice versa.

(c) $y = 2x + 5$ and $y = \frac{1}{2x+5}$



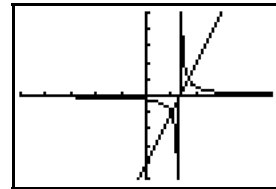
The graphs intersect at $(-2, 1)$ and $(-3, -1)$. These functions have been translated 2.5 units left and stretched horizontally by a factor of $\frac{1}{2}$, compared to the functions $y = x$ and $y = \frac{1}{x}$.

Alternatively, solve the system of equations to get the x -values of the points of intersection.

$$\begin{aligned} 2x + 5 &= \frac{1}{2x + 5} \\ (2x + 5)^2 &= 1 \\ 2x + 5 &= \pm 1 \end{aligned}$$

The zero for $y = 2x + 5$ occurs at $(-2.5, 0)$ and the asymptote is $x = -2.5$. When the original function is close to the x -axis, the reciprocal is far away and vice versa.

(d) $y = 3x - 4$ and $y = \frac{1}{3x-4}$



The graphs intersect at $(\frac{5}{3}, 1)$ and $(1, -1)$. These functions have been translated $\frac{4}{3}$ units right and stretched horizontally by a factor of $\frac{1}{3}$, compared to the functions

$y = x$ and $y = \frac{1}{x}$. Alternatively, solve the system of equations to get the x -values of the points of intersection to get the x -values for the points of intersection.

$$\begin{aligned} 3x - 4 &= \frac{1}{3x - 4} \\ (3x - 4)^2 &= 1 \\ 3x - 4 &= \pm 1 \\ 3x - 4 &= 1 \quad \text{and} \quad 3x - 4 = -1 \\ 3x &= 5 \quad \quad \quad 3x &= 3 \\ x &= \frac{5}{3} \quad \quad \quad x &= 1 \end{aligned}$$

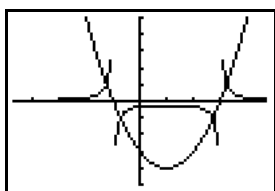
The zero for $y = 3x - 4$ occurs at $(\frac{4}{3}, 0)$ and the asymptote is $x = \frac{4}{3}$. When the original function is close to the x -axis, the reciprocal is far away and vice versa.

5. On paper, sketch the graph of $y = 2x - 1$. Using your answers for steps 1 to 4 as a guide, sketch the graph of $y = \frac{1}{2x-1}$. Use a calculator to verify your sketch.

Students will be able to sketch the functions if they follow the steps in this *Think, Do, Discuss*.

6. On a graphing calculator, change the window ($Xmin=-4.7$, $Xmax=4.7$, $Ymin=-5$, and $Ymax=5$). Graph the functions $y = (x-3)(x-1)$ and $y = \frac{1}{(x-3)(x+1)}$ on the same axes. You will need an extra pair of brackets around the product in the denominator of the reciprocal function: $Y2=1/((X-3)(X+1))$.

(a) At what points do the graphs intersect?



Use TRACE to find the intersection points or note that the x -coordinates of the points of intersection are slightly above and below the zeros of the original function. Solving algebraically,

$$x^2 - 2x - 3 = \frac{1}{x^2 - 2x - 3}$$

$$(x^2 - 2x - 3)^2 = 1$$

$$x^2 - 2x - 3 = \pm 1$$

$$x^2 - 2x - 3 = 1 \quad \text{and} \quad x^2 - 2x - 3 = -1$$

$$x^2 - 2x - 4 = 0 \quad \quad \quad x^2 - 2x - 2 = 0$$

For $x^2 - 2x - 4 = 0$, $a = 1$, $b = -2$, and $c = -4$.

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$

$$= 1 \pm \sqrt{5}$$

For $x = 1 + \sqrt{5}$,

$$\begin{aligned} y &= (1 + \sqrt{5} - 3)(1 + \sqrt{5} + 1) \\ &= (-2 + \sqrt{5})(2 + \sqrt{5}) \\ &= -4 + 5 \\ &= 1 \end{aligned}$$

One point of intersection is $(1 + \sqrt{5}, 1)$, or $(3.4, 1)$.

For $x = 1 - \sqrt{5}$,

$$\begin{aligned} y &= (1 - \sqrt{5} - 3)(1 - \sqrt{5} + 1) \\ &= (-2 - \sqrt{5})(2 - \sqrt{5}) \\ &= -4 + 5 \\ &= 1 \end{aligned}$$

Another point of intersection is $(1 - \sqrt{5}, 1)$, or $(-1.4, 1)$.

For $x^2 - 2x - 2 = 0$, $a = 1$, $b = -2$, and $c = -2$.

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= 1 \pm \sqrt{3} \end{aligned}$$

For $x = 1 + \sqrt{3}$,

$$\begin{aligned} y &= (1 + \sqrt{3} - 3)(1 + \sqrt{3} + 1) \\ &= (-2 + \sqrt{3})(2 + \sqrt{3}) \\ &= -4 + 3 \\ &= -1 \end{aligned}$$

Another point of intersection is $(1 + \sqrt{3}, -1)$ or $(2.7, -1)$.

Similarly, for $x = 1 - \sqrt{3}$, $y = -1$.

The last point of intersection is $(1 - \sqrt{3}, -1)$, or $(-0.7, -1)$.

- (b) What features of the graph of the reciprocal function exist at the zeros of the quadratic function?

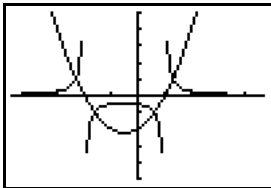
The asymptotes of the reciprocal function, $x = 3$ and $x = -1$, are at the zeros of the quadratic function.

- (c) Compare the shape of the graph of the reciprocal function with the shape of the parabola. Consider the lines of symmetry, the quadrants that each graph occupies, and the position of each graph.

Both graphs have a vertical line of symmetry at $x = 1$, halfway between the zeros of the original function. Both graphs also have points in all quadrants. In the first quadrant, the x -values of points for the original function are the same as those for the reciprocal function. Similar statements can be made for each quadrant.

7. Repeat step 6 for each of the following pairs of functions.

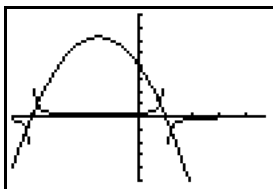
(a) $y = (x + 2)(x - 1)$, $y = \frac{1}{(x + 2)(x - 1)}$



Points of intersection are (1.3, 1), (-2.3, 1), (0.6, -1), and (-1.6, -1).

The asymptotes of the reciprocal function, $x = 1$ and $x = -2$, are at the zeros of the original function. Both graphs have a vertical line of symmetry at $x = -0.5$, halfway between the zeros of the original function. Both graphs have points in all quadrants. In the first quadrant, the x -values of points for the original function are the same as those for the reciprocal function. Similar statements can be made for each quadrant.

(b) $y = (1 - x)(4 + x)$, $y = \frac{1}{(1 - x)(4 + x)}$

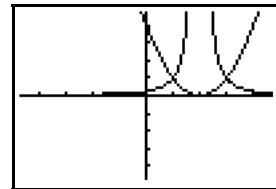


Points of intersection are (-3.8, 1), (0.8, 1), (-4.2, -1), and (1.2, -1).

The asymptotes of the reciprocal function, $x = -4$ and $x = 1$, are at the zeros of the original function. Both graphs have a vertical

line of symmetry at $x = -1.5$, halfway between the zeros of the original function. Both graphs have points in all quadrants. In the first quadrant, the x -values of points for the original function are the same as those for the reciprocal function. Similar statements can be made for each quadrant.

(c) $y = (x - 2)^2$, $y = \frac{1}{(x - 2)^2}$



Points of intersection are (1, 1) and (3, 1). There is one asymptote, $x = 2$, and both graphs are symmetrical about $x = 2$. All y -values for both the original and reciprocal functions are positive. In the first quadrant, the x -values of points for the original function are the same as those for the reciprocal function.

8. Sketch the parabola for the function $y = x^2 + x - 6$. First factor. Where are the vertical asymptotes of the graph of $y = \frac{1}{x^2 + x - 6}$? Draw the asymptotes.

$y = (x + 3)(x - 2)$

The parabola will open up with zeros at $x = -3$ and $x = 2$, which are also the equations for the asymptotes of the reciprocal function.

9. What points will the two graphs in question 8 have in common? Mark these points on the sketch. For the graph above the x -axis, decide what will happen to the graph of the reciprocal function to the right and left of the common points. Use the distance of the parabola from the x -axis as a guide. Sketch those parts of the graph for $y = \frac{1}{x^2 + x - 6}$.

The graphs will intersect where $x^2 + x - 6 = \pm 1$.

The intersection points are (2.2, 1), (-3.2, 1), (1.8, -1), and (-2.8, -1).

- 10. Determine the coordinates of the vertex. Use the y-coordinate of the vertex to find the value of $\frac{1}{x^2 + x - 6}$ corresponding to this y-value. Mark this point on the sketch. Then decide what will happen to the graph of the reciprocal function on each side of the common points below the x-axis. Sketch this final part of the graph for $y = \frac{1}{x^2 + x - 6}$.**
- Use a graphing calculator to verify your sketch

Additional Example: Functions and Relations or Functions

For a particular route, an airline determines that the income required to assure a satisfactory profit and cover costs of staff, fuel, meals, and so on is \$32 376.

- (a) Write an equation for the individual ticket price, t , a function of p , the number of expected passengers.
- (b) Choose a reasonable number of passengers and determine the price per ticket.
- (c) Determine the price per ticket if half as many passengers regularly use the flight.

What restrictions might the airline impose?

Solution

- (a) $t = \frac{32\,376}{p}$
- (b) If there are 142 passengers, the price will be \$228 per ticket. If there are 250 passengers, the price will be \$129.50 per ticket.
- (c) If only 71 passengers use the flight, the price would be \$456 per ticket. Note that, if the number of passengers is halved, the price doubles.

The airline might cancel any flight with fewer than 100 passengers and, of course, the size of the plane will determine an upper limit on the number of passengers.

ASSESSMENT SUGGESTION

Application

Use the related generic rubric (see Assessment and Evaluation in the *Teacher Resource*) to assess students' abilities to fit a reciprocal function to an appropriate situation.

ASSESSMENT SUGGESTION

Communication

Ask students to describe a situation that could be modelled using a reciprocal function.

Practise, Apply, Solve 4.6

Questions with a Functions and Relations Focus:

1–5, 7, 8, 10, 12, 14–19

Questions with a Functions Focus:

1–5, 7, 8, 10, 11, 13, 14, 16, 18, 19

10. Knowledge and Understanding, p. 347

Management Suggestion

Do (a) together as a class. Allow time for students to try (b) on their own and provide immediate feedback by showing the solution on the overhead.

Achievement Chart Connection

Students should demonstrate understanding of a reciprocal function and its domain and range and need to differentiate inverses from reciprocals.

Level 3 Performance

Students should show the reciprocal function as a section of a hyperbola and refer to the asymptotes when stating the domain and range.

14. Communication, p. 347

Management Suggestion

Give each student a different but similar equation. Invite selected students to present their work orally with the expectation that another student will be asked to judge the accuracy of the solution.

Achievement Chart Connection

Students must clearly explain mathematical reasoning.

Level 3 Performance

Students should identify zeros of the original function, connect zeros to asymptotes, and use appropriate notation to distinguish the original function from the reciprocal function.

16. Check Your Understanding, p. 347**Management Suggestion**

Invite students to discuss their solutions with a partner. As a whole class, develop a model solution on the overhead.

18. Thinking, Inquiry, Problem Solving, p. 348**Management Suggestion**

Invite students to submit individual solutions to include in a math portfolio. Expect evidence of organization and initiative.

Achievement Chart Connection

Students follow a relatively complex argument and draw conclusions from a suggested pattern.

Level 3 Performance

Students should complete the table, use the given pattern to fill in the blanks, extend the pattern, and use appropriate facts to estimate areas.

19. Application, p. 348**Management Suggestion**

Use two overheads, one with the given curve and another with this curve but no grid so that the curve can be easily translated according to students' suggestions.

Achievement Chart Connection

Students use known concepts and apply translations in an unfamiliar setting.

Level 3 Performance

Students should recognize the vertical and horizontal asymptotes and apply appropriate translations.