

4.11 Adding and Subtracting Rational Expressions

Specific Expectations

- OC1.07U add, subtract, multiply, and divide rational expressions, and state the restrictions on the variable values

Evidence of Learning

Students should be able to

- determine the lowest common denominator and state the restrictions for two or more rational expressions by first factoring all the denominators
- add and subtract rational expressions by first converting them to equivalent expressions with a common denominator
- analyze and then use a table to organize information
- solve problems involving inversely related variables by creating and solving equations containing rational expressions

Text Page 364

Time Required Functions and Relations:

75 min

Time Required Functions: 150 min

Materials and Equipment

- notepaper

Overview

- In the context of a distance-speed-time problem, add rational expressions.
- Extend adding and subtracting rational numbers for working with rational expressions.

Adding and Subtracting Rational Expressions

Students will need the skills that they will learn in this section for future work with limits, derivatives, and rational functions in the Advanced Functions and Introductory Calculus Functions course, as well as for proving by mathematical induction in the Geometry and Discrete Mathematics Functions and Relations course. The skills here are equally important for students in the Functions and Relations and Functions courses.

MANAGEMENT SUGGESTION

Review the relations between distance, speed, and time and the meaning of upstream and downstream before students begin Part 1. Also review finding the lowest common denominator and methods for adding fractions for Part 2.

Management Options

Groups – Pair students to work on the *Think, Do, Discuss* questions in Part 1. Take up the answers to these questions in a whole-class discussion. Then have pairs work on the *Think, Do, Discuss* questions in Part 2. Divide the board into two “columns” and have pairs of students write their answers to questions 1 to 5 in these columns. Take up these answers as a class before students do questions 6 to 8.

Teacher Demonstration – Discuss the *Think, Do, Discuss* questions in Part 1 with the class and have a student record answers on the board or overhead transparency. Divide the board into two columns and have two students do *Think, Do, Discuss* questions 1 to 5 in Part 2 following directions from the class. Demonstrate solutions to questions 6 to 8.

Independent Work – If you review “upstream,” “downstream,” and “LCD” ahead of time, assign the *Think, Do, Discuss* questions in parts 1 and 2 as independent work.

EXTENSION

Ask students to consider the following questions:

- How many numbers fit the equation $\frac{x}{y} + \frac{y}{x} = 2$?

Hint: Add the expressions on the left side and rewrite the equation as a quadratic relation.

- Solve the system.

$$\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 1 \\ 2x - 3y &= 4 \end{aligned}$$

Part 1: Investigating Rowing Speed

Think, Do, Discuss, text page 364

- How does the flow of the river affect the boat when it is being rowed upstream? downstream?

When the boat is being rowed upstream, it is moving against the current. The river is flowing in the opposite direction to the movement of the boat. The current will tend to slow down the boat's speed.

When the boat is being rowed downstream, the river and the boat are moving in the same direction. The current will tend to speed up the boat's speed.

- Suppose the crew's rowing speed in still water is x km/h. Write an expression for the upstream rowing speed.

$$(x - 4) \text{ km/h}$$

- If you know the distance and the speed, then how can you calculate time? Write an expression for the time the crew takes to row upstream.

$$\text{distance} = \text{speed} \times \text{time}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

The time taken to row upstream is $\frac{5}{x - 4}$ h.

- Now write an expression for the downstream rowing speed. Then write an expression for the time taken by the crew to row downstream.

The downstream speed is $(x + 4)$ km/h.

The time taken to row downstream is $\frac{5}{x + 4}$ h.

- How long did the round trip take? Write an equation that connects your answers to steps 3 and 4. Make sure that the values in the equation use the same units. To determine the rowing speed on still water, you will need to solve this equation.

The round trip took 40 min, or $\frac{2}{3}$ h.

$$\text{So } \frac{5}{x - 4} + \frac{5}{x + 4} = \frac{2}{3}.$$

Part 2: Adding and Subtracting Rational Expressions

Think, Do, Discuss, text page 365

- Make two columns on a piece of paper. At the top of the left column write "Rational Numbers" and, at the top of the right column, write "Rational Expressions." In the left column add $\frac{2}{7} + \frac{3}{7}$. In the right column, use the

same method to add $\frac{2}{x} + \frac{3}{x}$. Try to

match the steps as you work. What is the same about the two sums? What must you state about x in the right column?

Rational Numbers	Rational Expressions
$\frac{2}{7} + \frac{3}{7} = \frac{2+3}{7}$	$\frac{2}{x} + \frac{3}{x} = \frac{2+3}{x}$
$= \frac{5}{7}$	$= \frac{5}{x}, x \neq 0$

2. In the left column, add $\frac{3}{8} + \frac{1}{8}$. Reduce the answer to lowest terms. In the right column, use the same method to add $\frac{a}{2b} + \frac{3a}{2b}$. Reduce this answer to lowest terms. Remember to state any restriction(s).

Rational Numbers	Rational Expressions
$\frac{3}{8} + \frac{1}{8} = \frac{3+1}{8}$ $= \frac{4}{8}$ $= \frac{1}{2}$	$\frac{a}{2b} + \frac{3a}{2b} = \frac{a+3a}{2b}$ $= \frac{4a}{2b}$ $= \frac{2a}{b}, b \neq 0$

3. In the left column, subtract $\frac{5}{6} - \frac{4}{3}$. Remember to use the lowest common denominator (LCD). In the right column, try to subtract $\frac{7}{4y} - \frac{5}{2y}$, using the same steps. What is the LCD in this case? What is the restriction?

Rational Numbers	Rational Expressions
$\frac{5}{6} - \frac{4}{3} = \frac{5}{6} - \frac{8}{6}$ $= \frac{5-8}{6}$ $= \frac{-3}{6}$ $= -\frac{1}{2}$	$\frac{7}{4y} - \frac{5}{2y} = \frac{7}{4y} - \frac{10}{4y}$ $= \frac{7-10}{4y}$ $= \frac{-3}{4y}$ $= -\frac{3}{4y}, y \neq 0$

The LCD in the first column is 6 and the LCD in the second column is $4y$.

4. Subtract $\frac{3}{4} - \frac{2}{3}$ in the left column. Show the steps for changing each fraction so that the denominators are common. Follow the same steps to subtract $\frac{3}{x} - \frac{2}{y}$. The result will not look simpler, but this form may be useful for some purposes.

Rational Numbers

$$\begin{aligned}\frac{3}{4} - \frac{2}{3} &= \frac{3}{4} \times \frac{3}{3} - \frac{2}{3} \times \frac{4}{4} \\ &= \frac{9}{12} - \frac{8}{12} \\ &= \frac{1}{12}\end{aligned}$$

Rational Expressions

$$\begin{aligned}\frac{3}{x} - \frac{2}{y} &= \frac{3}{x} \times \frac{y}{y} - \frac{2}{y} \times \frac{x}{x} \\ &= \frac{3y}{xy} - \frac{2x}{xy} \\ &= \frac{3y-2x}{xy}, x \neq 0, y \neq 0\end{aligned}$$

5. To find the LCD for the sum $\frac{4}{15} + \frac{7}{10}$, factor each of the denominators. Express 15 and 10 as products of prime factors. Write the LCD as a product of prime factors. Find the sum in the left column. Use the same steps to add $\frac{4}{st} + \frac{7}{rt}$ in the right column.

Rational Numbers

$$15 = 3 \times 5 \text{ and } 10 = 2 \times 5$$

The LCD is $2 \times 3 \times 5 = 30$.

$$\begin{aligned}\frac{4}{15} + \frac{7}{10} &= \frac{4}{15} \times \frac{2}{2} + \frac{7}{10} \times \frac{3}{3} \\ &= \frac{8}{30} + \frac{21}{30} \\ &= \frac{29}{30}\end{aligned}$$

Rational Expressions

$$st = s \times t \text{ and } rt = r \times t$$

The LCD is $r \times s \times t = rst$.

$$\begin{aligned}\frac{4}{st} + \frac{7}{rt} &= \frac{4}{st} \times \frac{r}{r} + \frac{7}{rt} \times \frac{s}{s} \\ &= \frac{4r}{rst} + \frac{7s}{rst} \\ &= \frac{4r+7s}{rst}, r \neq 0, s \neq 0\end{aligned}$$

6. Simplify $\frac{a}{ab+b^2} - \frac{b}{a^2+ab}$. Use the method in step 5. Begin by factoring the denominators. What is the LCD? By what factor do you multiply $\frac{a}{b(a+b)}$ so that it will have the common denominator? Do the same for $\frac{b}{a(a+b)}$. Now that the expressions have the same denominators, find the difference. Notice that the resulting numerator can be factored. Factor the numerator and simplify the expression. Did you remember to state restrictions?

$$\frac{a}{ab+b^2} - \frac{b}{a^2+ab} = \frac{a}{b(a+b)} - \frac{b}{a(a+b)}$$

The LCD is $ab(a+b)$.

$$\begin{aligned}\frac{a}{b(a+b)} - \frac{b}{a(a+b)} \\ &= \frac{a}{b(a+b)} \times \frac{a}{a} - \frac{b}{a(a+b)} \times \frac{b}{b} \\ &= \frac{a^2-b^2}{ab(a+b)} \\ &= \frac{(a-b)(a+b)}{ab(a+b)}, a \neq 0, b \neq 0, a \neq -b \\ &= \frac{a-b}{ab}\end{aligned}$$

7. In Part 1, you should have obtained the equation $\frac{5}{x-4} + \frac{5}{x+4} = \frac{2}{3}$. To add the two rational expressions on the left side of this equation, follow the method in step 4. Treat each binomial as a single variable. What is

the common denominator? By what factor do you multiply $\frac{5}{x-4}$ so that the denominator will be the common denominator? Do the same for $\frac{5}{x+4}$. Add the numerators and simplify.

The LCD for the left side is $(x-4)(x+4)$.

$$\begin{aligned}\frac{5}{x-4} + \frac{5}{x+4} \\ &= \frac{5}{x-4} \times \frac{x+4}{x+4} + \frac{5}{x+4} \times \frac{x-4}{x-4} \\ &= \frac{5(x+4)+5(x-4)}{(x-4)(x+4)} \\ &= \frac{5x+20+5x-20}{x^2-16} \\ &= \frac{10x}{x^2-16}\end{aligned}$$

So the equation is $\frac{10x}{x^2-16} = \frac{2}{3}$.

8. You should now have the equation $\frac{10x}{x^2-16} = \frac{2}{3}$. Multiply both sides by (x^2-16) . Then multiply both sides by 3. Rearrange and solve the resulting quadratic equation. What was the crew's rowing speed in still water?

$$\frac{10x}{x^2-16} \times 3(x^2-16) = \frac{2}{3} \times 3(x^2-16)$$

$$30x = 2(x^2-16)$$

$$2x^2 - 30x - 32 = 0$$

$$2(x^2 - 15x - 16) = 0$$

$$2(x-16)(x+1) = 0$$

$$x = 16 \text{ and } x = -1$$

Reject the negative root since $x > 0$. The crew's rowing speed in still water was 16 km/h.

Additional Example: Functions and Relations

Show that $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

Solution

L.S.	R.S.
$\begin{aligned} & \frac{k}{k+1} \times \frac{k+2}{k+2} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)(k+1)}{(k+1)(k+2)}, k \neq -1, -2 \\ &= \frac{k+1}{k+2} \end{aligned}$	$\frac{k+1}{k+2}$

Additional Example: Functions

Simplify $\frac{1}{x+h} - \frac{1}{x}$ and state the result in lowest terms.

Solution

$$\frac{1}{x+h} - \frac{1}{x} = \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right)$$

The restrictions are $x \neq 0$, $h \neq 0$, and $x \neq -h$.

Use the usual order of operations and do the operations in the brackets first.

Within the brackets, the lowest common denominator is $x(x+h)$.

$$\begin{aligned} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right) &= \frac{1}{h} \left(\frac{1}{x+h} \times \frac{x}{x} - \frac{1}{x} \times \frac{x+h}{x+h} \right) \\ &= \frac{1}{h} \left[\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)} \right] \\ &= \frac{1}{h} \left[\frac{x - x - h}{x(x+h)} \right] \\ &= \frac{1}{h} \left[\frac{-h}{x(x+h)} \right] \\ &= \frac{-1}{x(x+h)} \end{aligned}$$

Additional Example: Functions and Relations or Functions

A landscape gardener agrees to do a project for \$1200. Because of rain, he takes three days longer than expected to complete the job, and his daily earnings are \$90 less than expected. In how many days did he expect to complete the project?

Solution

Let x represent the number of days the landscaper expected the project.

	Total Earnings (\$)	Number of Days to Complete Job	Amount Earned per Day
Expected	1200	x	$\frac{1200}{x}$
Actual	1200	$x+3$	$\frac{1200}{x+3}$

Since the landscaper earned \$90 less per day than he expected,

$$\frac{1200}{x} - \frac{1200}{x+3} = 90, x \neq -3, 0$$

Multiply by the LCD, $x(x+3)$.

$$\begin{aligned} 90 \times x(x+3) &= \left(\frac{1200}{x} \right) \times x(x+3) - \left(\frac{1200}{x+3} \right) \times x(x+3) \\ 90x^2 + 270x &= 1200(x+3) - 1200x \\ 90x^2 + 270x &= 3600 \\ 90x^2 + 270x - 3600 &= 0 \\ 90(x^2 + 3x - 40) &= 0 \\ 90(x+8)(x-5) &= 0 \\ x = -8 \text{ or } x = 5 \end{aligned}$$

Reject the negative root, since $x > 0$.

The landscape gardener had expected to complete the project in five days.

Practice, Apply, Solve 4.11

Questions with a Functions and Relations

Focus:

4–13, 15, 16, 18–21

Questions with a Functions Focus:

1–9, 11, 12, 15, 16, 18, 19

7. Communication, p. 370

Management Suggestion

Have students discuss the question in pairs or small groups before they write individual responses.

Achievement Chart Connection

Students explain their reasons in writing, or orally, and use mathematical language and symbols.

Level 3 Performance

When adding or subtracting rational expressions, find the LCD instead of just multiplying the denominators to reduce the number and degree of terms in both the numerator and denominator of the sum or difference.

For example,

$$\frac{x}{x^2-4} - \frac{3x}{3x^2+5x-2}$$
$$= \frac{x}{(x-2)(x+2)} - \frac{3x}{(3x-1)(x+2)}, x \neq \frac{1}{3}, \pm 2$$

Just multiplying the denominators yields a common denominator that is a product of binomials.

$$\frac{x}{(x-2)(x+2)} \times \frac{(3x-1)(x+2)}{(3x-1)(x+2)}$$
$$- \frac{3x}{(3x-1)(x+2)} \times \frac{(x-2)(x+2)}{(x-2)(x+2)}$$

A better way is to find the LCD. Since the factor $(x+2)$ is common to both denominators, the LCD is $(x-2)(x+2)(3x-1)$.

Then

$$\frac{x}{(x-2)(x+2)} \times \frac{3x-1}{3x-1} - \frac{3x}{(3x-1)(x+2)} \times \frac{x-2}{x-2}$$

This expression is easier to work with.

8. Knowledge and Understanding, p. 370

Management Suggestion

Have students work individually on this question. Circulate and note any problems they may have. When most students seem to have finished, have them share their solutions in groups of four. Invite two students to write the solutions to the two parts on the board or on an overhead transparency.

Achievement Chart Connection

Students find the LCD and add and subtract rational expressions.

ASSESSMENT LEARNING SKILLS

Teamwork

Observe groups of students as they work. Are all group members contributing to the discussion? Are they working well as a group?

Level 3 Performance

Students should show all steps and state restrictions.

(a)

$$\frac{5x-7y}{12x} + \frac{2x-9y}{8y}, x \neq 0, y \neq 0$$
$$= \frac{5x-7y}{12x} \times \frac{2y}{2y} + \frac{2x-9y}{8y} \times \frac{3x}{3x}$$
$$= \frac{2y(5x-7y)}{24xy} + \frac{3x(2x-9y)}{24xy}$$
$$= \frac{10xy-14y^2+6x^2-27xy}{24xy}$$
$$= \frac{6x^2-17xy-14y^2}{24xy} \text{ or } \frac{(3x+2y)(2x-7y)}{24xy}$$

(b)

$$\begin{aligned}& \frac{1}{2x+8} - \frac{3x}{(x+4)^2} + \frac{1}{2} \\&= \frac{1}{2(x+4)} - \frac{3x}{(x+4)^2} + \frac{1}{2}, x \neq -4 \\&= \frac{1}{2(x+4)} \times \frac{(x+4)}{(x+4)} - \frac{3x}{(x+4)^2} \times \frac{2}{2} + \frac{1}{2} \times \frac{(x+4)^2}{(x+4)^2} \\&= \frac{(x+4)}{2(x+4)^2} - \frac{6x}{2(x+4)^2} + \frac{(x+4)^2}{2(x+4)^2} \\&= \frac{x+4-6x+x^2+8x+16}{2(x+4)^2} \\&= \frac{x^2+3x+20}{2(x+4)^2}\end{aligned}$$

15. Application, p. 371

Management Suggestion

Have students work in small groups. Collect one complete solution per group for assessment.

Achievement Chart Connection

Students must recognize that the number of T-shirts, the price per shirt, and the total cost are related in the same way that distance, speed, and time are related in other questions. They can apply a similar model to solve this problem.

Level 3 Performance

Students should show all steps, explain any variables, and state restrictions on these variables. They may organize the information in a table or argue as follows.

Let x be the number of T-shirts in a case. If Jaime paid \$450 for x T-shirts, each one cost $\frac{450}{x}$. Since she kept two for herself, she sold $(x-2)$ shirts. If she sold $(x-2)$ shirts for \$560, then each shirt cost $\frac{560}{x-2}$. She made a profit of \$10 on each T-shirt, so the selling price must have been \$10 more than the cost.

$$\frac{560}{x-2} - \frac{450}{x} = 10, x \neq 0, 2$$

Multiply by the LCD, which is $x(x-2)$.

$$\left(\frac{560}{x-2}\right) \times x(x-2) - \left(\frac{450}{x}\right) \times x(x-2) = 10 \times x(x-2)$$

$$560x - 450(x-2) = 10x^2 - 20x$$

$$560x - 450x + 900 = 10x^2 - 20x$$

$$110x + 900 = 10x^2 - 20x$$

$$10x^2 - 130x - 900 = 0$$

$$10(x-18)(x+5) = 0$$

$$x = 18 \text{ or } x = -5$$

Reject the negative root, since $x > 2$.

There were 18 shirts in the case.

18. Check Your Understanding, p. 371

Management Suggestion

In pairs, students can take turns to write alternate steps. Use this question as a warm-up activity or homework to be checked the next day after it is assigned.

19. Thinking, Inquiry, Problem

Solving, p. 371

Management Suggestion

Students will find it challenging to develop an algebraic solution. Do not expect them to solve it quickly. Assign it as a "Problem of the Week" and encourage students to persevere.

Achievement Chart Connection

Students must add the rational expressions on the right side, compare the original and the new expressions, and develop a method to evaluate A and B from this new information.

EXTENSION SUGGESTION

Students in the Functions and Relations course might benefit from practising adding and subtracting rational expressions since they will encounter this topic in mathematical induction in the Geometry and Discrete Mathematics course in grade 12.

Level 3 Performance

Check for sound reasoning. Students must state restrictions and be aware of them throughout the solution.

$$\frac{2x-1}{(x+1)(3x+2)} = \frac{A}{x+1} + \frac{B}{3x+2}, x \neq -1, -\frac{2}{3}$$

Find A and B by adding the expressions on the right side and comparing numerators on each side.

$$\begin{aligned} \frac{A}{(x+1)} \times \frac{(3x+2)}{(3x+2)} + \frac{B}{(3x+2)} \times \frac{(x+1)}{(x+1)} \\ = \frac{A(3x+2)}{(x+1)(3x+2)} + \frac{B(x+1)}{(3x+2)(x+1)} \\ = \frac{A(3x+2) + B(x+1)}{(x+1)(3x+2)} \end{aligned}$$

$$\text{So } \frac{2x-1}{(x+1)(3x+2)} = \frac{A(3x+2) + B(x+1)}{(x+1)(3x+2)}.$$

Since the denominators are the same, the numerators must also be the same.

$$2x - 1 = A(3x + 2) + B(x + 1)$$

$$2x - 1 = 3Ax + 2A + Bx + B$$

$$2x - 1 = (3A + B)x + 2A + B$$

Compare coefficients of x : $2 = 3A + B$ ①

Compare constant terms: $-1 = 2A + B$ ②

Subtract: $3 = A$ ① - ②

Substitute $A = 3$ in ①: $2 = 3(3) + B$

$$-7 = B$$

The Chapter Problem—Fundraising

In this section and sections 4.9 and 4.10, students have developed some purely algebraic skills they will need when dealing with rational functions next year.

The Chapter Problem Answers are on p. 4-63 of the *Teacher Resource*.

Challenge 1

Remind students that they should be able to

- add or subtract a reciprocal function and a linear function to create a function with an oblique asymptote. (The functions in *Practise, Apply, Solve* question 10 are examples. Students could graph these functions.)
- find the reciprocal of a square root function to get a graph in only two regions of the x - y plane

Challenge 2

Remind students that the area function is not a rational function since it contains a square root. However, they can graph this function and determine the height for which the area is the smallest.

The Challenge Answers are on p. 4-68 of the *Teacher Resource*.