

5.3 Trigonometric Functions

Specific Expectations

The following expectations are common to both the Functions and Relations course and the Functions course.

- TF1.01 determine the sine, cosine, and tangent of angles greater than 90° , using a suitable technique (e.g., related angles, the unit circle), and determine two angles that correspond to a given single trigonometric function value
- TF2.06 solve linear trigonometric equations
- TF3.01 sketch the graphs of $y = \sin x$ and $y = \cos x$, and describe their periodic properties
- TF3.05 write the equation of a sinusoidal function, given its graph and given its properties
- TF3.06 sketch the graph of $y = \tan x$; identify the period, domain, and range of the function; and explain the occurrence of asymptotes
- TF4.01 determine, through investigation, the periodic properties of various models (e.g., the table of values, the graph, the equation) of sinusoidal functions drawn from a variety of applications

Evidence of Learning

Students should be able to

- develop a table and determine the equation that represents the table
- define the primary trigonometric functions for a point $P(x, y)$ on the terminal arm of angle θ in standard position
- graph $y = \sin \theta$, $y = \cos \theta$, and $y = \tan \theta$, where θ is measured in degrees
- identify the properties of the graphs of the primary trigonometric functions
- solve a simple linear trigonometric equation for θ within a specified domain

Text Page 424

Time Required Functions and Relations or Functions: 150 min

Materials and Equipment

- graph paper, ruler

Overview

- Introduce the primary trigonometric functions.
- Graph the primary trigonometric functions and examine the properties of their graphs.
- Develop the CAST rule.
- Relate key points on the unit circle to the graphs of the primary trigonometric functions.

Trigonometric Functions

This section introduces students to the graphs of $y = \sin \theta$, $y = \cos \theta$ and $y = \tan \theta$, where θ is an angle in degrees. In Part 1, the rotation of a seat on a Ferris wheel introduces the rotation of a point $P(x, y)$ about the centre of a circle.

In Part 2, the primary trigonometric ratios of right triangles are extended to any point $P(x, y)$ on a circle of radius r and rotated through angle θ .

In Part 3, students will discover that the position of $P(x, y)$ is a function of time by linking the angle of rotation to time.

The section concludes with students developing the CAST rule, which will help them to remember in which quadrants the primary trigonometric ratios are positive.

Management Options

Groups – Arrange students into groups of four. Then each group can do the *Think, Do, Discuss* questions in Part 1. Monitor each group to ensure that all questions are discussed. Introduce the *Think, Do, Discuss* of Part 2 using the board or overhead. Assign question 1 to half the groups and question 2 to the other half. Demonstrate question 3, using the groups' responses to questions 1 and 2. Graph the tangent function as a whole class activity.

Groups can answer the *Think, Do, Discuss* questions of Part 3. Consider giving a prize to the first and second groups who complete the tasks correctly.

Demonstrate the *Think, Do, Discuss* in Part 4.

Teacher Demonstration – Lead students through Part 1. Put the introduction to Part 2 on the board or overhead. Continue leading students through the rest of Part 2, including the graphing of the tangent function. Have pairs brainstorm responses to *Think, Do, Discuss* questions 1 and 2 in Part 3. Answer questions 3, 4, and 5 as a whole class activity.

Develop the CAST rule at the board or overhead by having various students answer the *Think, Do, Discuss* questions in Part 4.

Independent Work – To introduce the lesson, assign all questions in the *Think, Do, Discuss* of Part 1 and question 1 of the *Think, Do, Discuss* of Part 2 for homework. Then demonstrate the rest of Part 2 the next day. Have students work through parts 3 and 4 on their own in class. Take up the results.

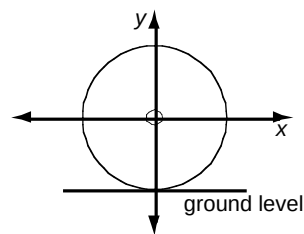
TECHNOLOGY SUGGESTION

Use a graphing calculator to draw the scatter plot for any table in this section. (Students could also use the sinusoidal regression operation of the TI-83 Plus, but this topic appears in section 5.9.)

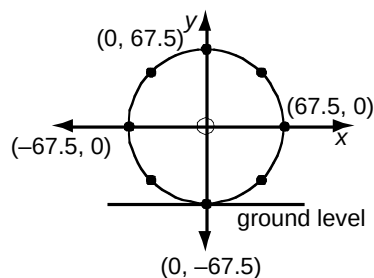
Part 1: Collecting Data for the Model

Think, Do, Discuss, text page 424

1. Draw a circle to model the London Eye. Centre the circle at the origin of the x - and y -axes. At the bottom of the circle, draw a horizontal line to represent the ground.



2. Draw points on the circle to indicate different positions on the ride. Choose points that are easy to estimate, such as points on the x -axis, the y -axis, and on the rotation angle halfway between the x - and y -axes in each quadrant.



- (a) How high above the ground will Sarah and Billy be when they are at the top of the ride? halfway to the top? Show these points on the diagram. What are the coordinates of each point? How can you use each coordinate to determine the height above the ground?

Sarah and Billy will be 135 m above the ground when they are at the top of the ride.

They will be 67.5 m above the ground when they are halfway to the top.

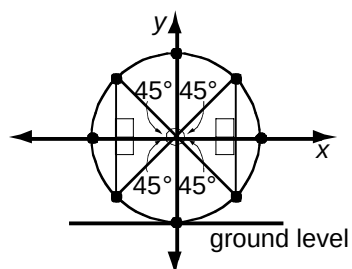
In the diagram, the circle is centred at the origin of the x - and y -axes (the centre is 67.5 m above the ground). The centre of this circle is $(0, 0)$ and the radius is 67.5 m. The points are, beginning at the positive x -axis, $(67.5, 0)$, $(0, 67.5)$, and $(-67.5, 0)$, because there are two halfway points.

The y -coordinate of each point relates to the height above the ground. Add 67.5 to each y -coordinate to get the height above the ground.

- (b) What acute angle corresponds to halfway between each quadrant? How many points on the ride will this angle help determine?

45° corresponds to halfway between each quadrant. There are four points.

- (c) Draw right triangles to the circle that include each related acute angle.



- (d) How long is the hypotenuse of each right triangle? Explain your answer.

The hypotenuse is the radius of the wheel, so the length of the hypotenuse is 67.5 m.

- (e) What trigonometric ratio combines the acute angle and the hypotenuse to give a second side of the triangle that represents vertical distance?

$$\sin \theta = \frac{\text{OPP}}{\text{HYP}}$$

- (f) How could you use this second side length to determine Sarah and Billy's location on the London Eye with respect to the x-axis? How could you use this second side length to find their height above the ground?

For each right triangle, the “second side length” is the side opposite the 45° angle and represents the y-coordinate. Above the x-axis, the y-coordinate is positive and below the x-axis, the y-coordinate is negative.

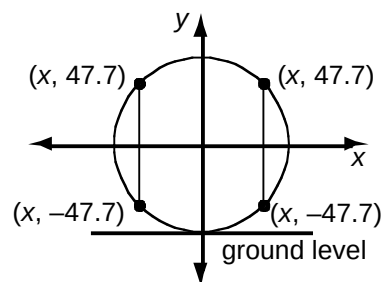
$$\sin 45^\circ = \frac{\text{OPP}}{67.5}$$

$$\text{OPP} = 67.5(\sin 45^\circ)$$

$$\text{OPP} \doteq 47.7$$

So the second side length is about 47.7 m long.

With respect to the x-axis, Sarah and Billy's locations that correspond to the related acute angles of 45° are $(x, 47.7)$, $(-x, 47.7)$, $(-x, -47.7)$, and $(x, -47.7)$.



At the first point, their height above the ground is $67.5 \text{ m} + 47.7 \text{ m}$, or 115.2 m.

At the second point, their height is also 115.2 m.

At the third point, their height is $67.5 \text{ m} - 47.7 \text{ m}$, or 19.8 m.

At the fourth point, their height is also 19.8 m.

- (g) How many different positions on the Ferris wheel have you represented on your diagram? What principal angles in standard position do these positions represent? By using the angles in standard position, what assumption have you made about where Billy and Sarah get on the ride?

Eight positions are represented on the diagram. However, there would be nine positions if Billy and Sarah return to the starting position.

The principal angles are 0° , 45° , 90° , 135° , 180° , 225° , 270° , 315° , 360° , assuming Billy and Sarah get on at 0° (the standard position).

- (h) Determine the height above the ground, to the nearest decimal, for each angle in standard position. Create a table in your notebook to record the data.

0°	45°	90°
67.5	115.2	135
135°	180°	225°
115.2	67.5	19.8
270°	315°	360°
0	19.8	67.5

Part 2: Extending the Primary Trigonometric Ratios Beyond Triangles

In this part, students relate the trigonometry of right triangles to the trigonometric functions.

Think, Do, Discuss, text page 426

1. (a) Which of the primary trigonometric functions best links the distance of a rider on the Ferris wheel from the horizontal diameter of the wheel, the size of the Ferris wheel, and the angle of rotation? Solve the equation for this distance in terms of the other two variables.

$$\sin \theta = \frac{y}{r}$$

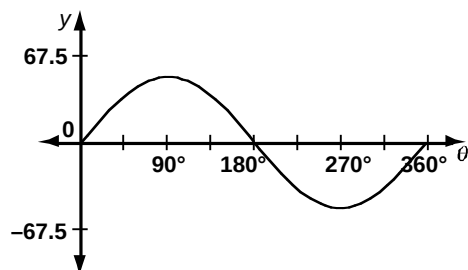
$$y = r \sin \theta$$

In this case, $y = 67.5 \sin \theta$.

- (b) Complete the table to show this distance for the London Eye as a function of angular rotation. Use increments of 30° for one complete revolution.

Angle of Rotation	0°	30°	60°	90°	120°
Distance on Ferris Wheel from the Horizontal Diameter of the Wheel (m)	0	33.8	58.5	67.5	58.5
Angle of Rotation	150°	180°	210°	240°	
Distance on Ferris Wheel from the Horizontal Diameter of the Wheel (m)	33.8	0	-33.8	-58.5	
Angle of Rotation	270°	300°	330°	360°	
Distance on Ferris Wheel from the Horizontal Diameter of the Wheel (m)	-67.5	-58.5	-33.8	0	

- (c) Graph the distance from the horizontal diameter versus the angle of rotation.



- (d) What is the greatest distance above the horizontal diameter? below the

horizontal diameter? How can these values be used to determine the amplitude of the curve?

The greatest distance above the horizontal axis is 67.5 m and, below the horizontal axis, 67.5 m. The amplitude is the distance from the axis of the curve to its maximum or minimum value. So the amplitude is 67.5.

- (e) What is the period of the curve?

The period is 360° .

- (f) How would the graph look for two revolutions of the Ferris wheel? What type of model would describe the data? What equation models the distance from the horizontal diameter for any angle of rotation for the Ferris wheel?

The graph repeats from 360° to 720° . A periodic model would describe this data. The equation for this model is

$$y = 67.5 \sin \theta.$$

- (g) What is Sarah and Billy's distance from the horizontal diameter when the ride has rotated 130° ? 280° ? 420° ? 550° ? What is their height above the ground for each of these positions of the Ferris wheel if it almost touches the ground?

$$\begin{aligned} y &= 67.5 \sin 130^\circ \\ &= 51.7, \text{ to one decimal place} \end{aligned}$$

Their height is $67.5 + 51.7$ or 119.2 m.

$$\begin{aligned} y &= 67.5 \sin 280^\circ \\ &= -66.5, \text{ to one decimal place} \end{aligned}$$

Their height is $67.5 - 66.5$ or 1 m.

$$\begin{aligned} y &= 67.5 \sin 420^\circ \\ &= 58.5, \text{ to one decimal place} \end{aligned}$$

Their height is $67.5 + 58.5$ or 126 m.

$$\begin{aligned} y &= 67.5 \sin 550^\circ \\ &= -11.8, \text{ to one decimal place} \end{aligned}$$

Their height is $67.5 - 11.8$ or 55.7 m.

- (h) Which coordinate is used to represent distance from the horizontal diameter?

The y-coordinate represents the distance from the horizontal diameter.

- (i) Write an equation to model their height above the ground for any angle of rotation of the Ferris wheel.

$$y = 67.5 \sin \theta + 67.5 \text{ or}$$

$$h(\theta) = 67.5 \sin \theta + 67.5$$

2. (a) Which primary trigonometric function best links the distance of a rider from the vertical diameter of the Ferris wheel, the size of the wheel, and the angle of rotation? Solve the equation for this distance in terms of the other two variables.

$$\cos \theta = \frac{x}{r}$$

$$x = r \cos \theta$$

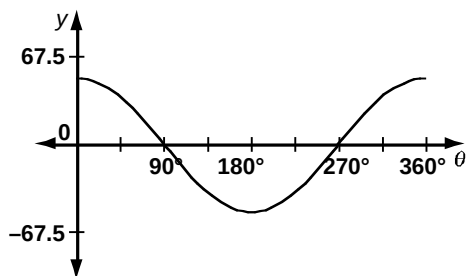
In this case, $x = 67.5 \cos \theta$.

- (b) Complete a table as in step 1(b) to show distance on the London Eye from the vertical diameter of the wheel for the given angles of rotation of the ride.

Angle of Rotation	0°	30°	60°	90°	120°
Distance on Ferris Wheel from the Vertical Diameter of the Wheel	67.5	58.5	33.8	0	-33.8

150°	180°	210°	240°	270°	300°	330°	360°
-58.5	-67.5	-58.5	-33.8	0	33.8	58.5	67.5

- (c) Graph the distance from the vertical diameter versus the angle of rotation.



- (d) What is the greatest horizontal distance to the right of the vertical diameter? to the left of the vertical diameter? How do these distances compare to those in step 1(d)? Explain. What is the amplitude of the curve?

The greatest distance to the left, or right, of the vertical diameter is 67.5 m. This distance is the same as the greatest distance above, or below, the horizontal diameter, because the rotation is about a circle. The amplitude of the curve is 67.5.

- (e) What is the period of the curve?

The period is 360°.

- (f) Extend the graph to represent two revolutions of the Ferris wheel.

The curve repeats from 360° to 720°.

- (g) What is Sarah and Billy's distance from the vertical diameter when the ride has rotated 130°? 280°? 420°? 550°?

$$x = 67.5 \cos 130^\circ$$

$$= -43.4, \text{ to one decimal place}$$

Their distance from the vertical diameter is 43.4 m, to the left of the diameter.

$$x = 67.5 \cos 280^\circ$$

$$= 11.8, \text{ to one decimal place}$$

Their distance from the vertical diameter is 11.8 m, to the right of the diameter.

$$x = 67.5 \cos 420^\circ$$

$$= 33.8, \text{ to one decimal place}$$

Their distance from the vertical diameter is 33.8 m, to the right of the diameter.

$$x = 67.5 \cos 550^\circ$$

$$= -66.5, \text{ to one decimal place}$$

Their distance from the vertical diameter is 66.5 m, to the left of the diameter.

- (h) Which coordinate is used to represent distance from the vertical diameter?

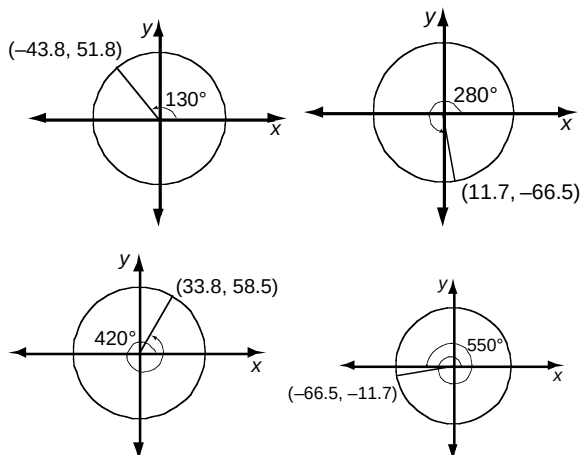
The x-coordinate represents the distance from the vertical diameter.

- (i) Write an equation to model their distance from the vertical diameter at any angle of rotation.

$$x = 67.5 \cos \theta \text{ or}$$

$$f(\theta) = 67.5 \cos \theta$$

3. (a) What x- and y-coordinates would represent Sarah and Billy's position on the London Eye at 130° ? 280° ? 420° ? 550° ? Show the coordinates on a diagram that models the Ferris wheel.



- (b) What general coordinates would represent their position on this Ferris wheel for any angle of rotation θ ? What coordinates would represent their position on any Ferris wheel or circular path of radius r ?

$$(x, y) = (67.5 \cos \theta, 67.5 \sin \theta)$$

$$\text{In general, } (x, y) = (r \cos \theta, r \sin \theta)$$

- (c) Draw a circle with $r = 1$ as a model of the Ferris wheel. Mark the exact coordinates of the eight positions you prepared in Part 1, step 2(g) if the Ferris wheel had a radius of 1.

At each halfway point, you can draw an isosceles, right triangle with angles 45° , 45° , and 90° . The hypotenuse of each triangle is 1. Let the length of each of the two equal sides be a .

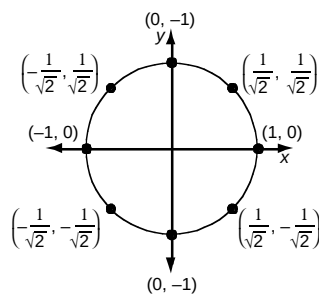
Then

$$a^2 + a^2 = 1$$

$$2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

$$a = \frac{1}{\sqrt{2}}$$



Graphing the Tangent Function

Think, Do, Discuss, text page 427

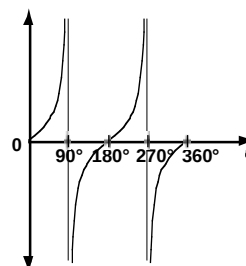
1. (a) Use the data from step 1(b) and step 2(b) above to make a table to represent the tangent function. How does this data apply to the definition of the tangent function? At what points on the Ferris wheel will the tangent function not be defined? What will happen to the graph at these points?

Angle of Rotation	0°	30°	60°	90°	120°
$\tan \theta$	0	0.57	1.73	—	-1.73
Angle of Rotation	150°	180°	210°	240°	
$\tan \theta$	-0.58	0	0.58	1.73	
Angle of Rotation	270°	300°	330°	360°	
$\tan \theta$	—	-1.73	-0.58	0	

The tangent function is the vertical distance from the horizontal axis to a point on the wheel divided by the distance from the vertical axis to the same point on the Ferris wheel.

The tangent function is not defined if the distance from the vertical axis, x , is 0, so there are asymptotes at 90° and 270° .

- (b) Graph the tangent function for one revolution of the Ferris wheel.



- (c) *What happens to the graph as the Ferris wheel starts its second revolution?*

As the Ferris wheel starts its second revolution, the graph continues to increase, and this increase can be related to the counterclockwise rotation of the Ferris wheel.

ASSESSING LEARNING SKILLS

Teamwork

When students are working in pairs or small groups, check to see how well students work together. Does each person contribute in a positive way that is noticed and recognized by others in the group? Use the Generic Rubric for Teamwork.

Part 3: Examining a Trigonometric Function in Terms of Time

Think, Do, Discuss, text page 427

1. *How many degrees does the Ferris wheel move every minute?*

360° divided by 30 min is $12^\circ/\text{min}$. The Ferris wheel rotates 12° every minute.

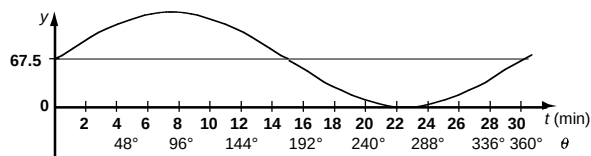
2. *Assume Billy and Sarah start the ride halfway to its maximum height. Make a table that shows their height every 2 min for one revolution of the ride.*

Time (min)	0	2	4	6
θ	0°	24°	48°	72°
Height (m)	67.5	95	117.7	131.7
Time (min)	8	10	12	14
θ	96°	120°	144°	168°
Height (m)	134.6	126	107.2	81.5
Time (min)	16	18	20	22
θ	192°	216°	240°	264°
Height (m)	53.5	27.8	9	0.4
Time (min)	24	26	28	30
θ	288°	312°	336°	360°
Height (m)	3.3	17.3	40	67.5

TECHNOLOGY SUGGESTION

Use a graphing calculator to draw the scatter plot of height versus time. Refer to this scatter plot when graphing height versus time.

3. *Graph height versus time.*



4. *Write an equation to model their height at any time, t .*

$$h(t) = 67.5 \sin(12t)^\circ + 67.5$$

5. *What is their height at 17 min?*

$$h(17) = 67.5 \sin[12(17)]^\circ + 67.5$$

$$= 81.1, \text{ to one decimal place}$$

At 17 min, their height is about 81.1 m.

Part 4: Developing the CAST Rule

Think, Do, Discuss, text page 428

1. *Draw a circle and mark one point in each quadrant that is on the circumference.*

Select any point in each quadrant that is clearly not on the x - or the y -axis.

2. *What is the sign of the x -coordinate? the y -coordinate?*

Quadrant I: $(x, y) = (+, +)$

Quadrant II: $(x, y) = (-, +)$

Quadrant III: $(x, y) = (-, -)$

Quadrant IV: $(x, y) = (+, -)$

3. *What is the sign of the radius?*

The radius is positive in any quadrant.

4. Determine the sign of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for each point.

	Q I	Q II	Q III	Q IV
$\sin \theta = \frac{y}{r}$	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{-}{+} = -$
$\cos \theta = \frac{x}{r}$	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{-}{+} = -$	$\frac{+}{+} = +$
$\tan \theta = \frac{y}{x}$	$\frac{+}{+} = +$	$\frac{+}{-} = -$	$\frac{-}{-} = +$	$\frac{-}{+} = -$

5. In which quadrants is
- (a) the sine function positive?
quadrants I and II
 - (b) the cosine function positive?
quadrants I and IV
 - (c) the tangent function positive?
quadrants I and III

Additional Example: Functions and Relations:

Determine the zeros of the function $y = \sin \theta$, $\theta = n^\circ$, $n \in \mathbf{R}$.

Solution

The function $y = \sin \theta$ for $0^\circ \leq \theta \leq 360^\circ$ has zeros at 0° , 180° , and 360° . The common difference is 180° . This pattern applies to any number of 360° periods. When $\theta = n^\circ$ and $n \in \mathbf{R}$, then there are zeros at $\theta = 180^\circ n$, $n \in \mathbf{I}$.

Additional Example: Functions

For $\sin \theta = -\frac{7}{8}$, $360^\circ \leq \theta \leq 0^\circ$, find all of the values of θ to the nearest degree.

Solution

$$\theta = \sin^{-1}\left(-\frac{7}{8}\right)$$

$$\theta \doteq -61^\circ$$

Next examine the related acute angle 61° , the domain $360^\circ \leq \theta \leq 0^\circ$, and the CAST rule.

The CAST rule indicates that $\sin \theta$ is negative in quadrants III and IV. So $\theta = -61^\circ$

(quadrant IV) and $\theta = 360^\circ - 61^\circ$, or 299° (quadrant IV).

Practise, Apply, Solve 5.3

8. Knowledge and Understanding, p. 435

Management Suggestion

The equation in the question is a trigonometric identity, which will be dealt with more formally in Chapter 6. Extend the question by asking students to select different points on the terminal arm and to verify the relation for these points. They should also verify the relation for different angles, using the memory key on a calculator to simplify calculations.

Achievement Chart Connection

Students need to know the primary trigonometric functions for a point on the terminal arm of an angle in standard position, and to substitute the values for x , y , and r for verifying the equation.

Level 3 Performance

Students should demonstrate understanding of the definitions of the primary trigonometric functions. To verify the equation, they must accurately simplify calculations.

12. Application, p. 435

Management Suggestion

Have students work in pairs. Direct one student to compute the answers and the other to sketch the graph of $y = \sin \alpha$ over the given domain. Sketch the line $y = 0.88$ and locate all points where the graph and the line intersect. Interpolate α -values. Students should verify results with a partner.

Achievement Chart Connection

Students apply the concept of a trigonometric function to solve for all angles over a specific domain.

Level 3 Performance

Students should accurately follow the procedure for determining all the angles in a given domain and for a given trigonometric function.

20. Communication, p. 436

Management Suggestion

Functions and Relations students could write the sequence of points on the θ -axis, -90° , 90° , 270° , 450° , 630° , using set notation in this way:

$$\{\theta \mid \theta = 90(2n - 1)^\circ, 0 \leq n \leq 4, n \in \mathbf{I}\}$$

Achievement Chart Connection

Students are expected to use mathematical language, symbols, visuals, and conventions for all tasks in this question.

Level 3 Performance

Students should clearly draw and label the graph of the tangent function for $-90^\circ \leq \theta \leq 630^\circ$, including the asymptotes. They should also write the sequence of points where the asymptotes cross the θ -axis in correct notation.

22. Check Your Understanding, p. 436

Management Suggestion

Have students write their answers in their notebooks. Then ask them to share with other students to find different ways to answer the question. Take up several versions and comment on the effectiveness of each version. Ask students to compare versions.

The graphs of $y = \sin \theta$ and $y = \cos \theta$ are sinusoidal. They each rise to a distinct maximum and fall to a distinct minimum. The graph of $y = \tan \theta$ has asymptotes. A curve approaches an asymptote, but does not cross it. For example, between 90° and 270° , the curve intersects the θ -axis at 180° . To the right of this point, θ approaches 270° and the y -values become increasingly large and positive. To the left of this point, θ approaches 90° and the y -values become increasingly large, but negative.

23. Thinking, Inquiry, Problem Solving, p. 437

Management Suggestion

Functions students may approach this question as an experiment. Students can form groups of three. One person can construct the 10-cm cube, while the other two decide how to conduct and record the experiment. Have string available, but do not pre-cut it. Let students decide the length of string needed, and answer questions as they arise.

Achievement Chart Connection

Students need to apply reasoning skills, because the function is not trigonometric but is still periodic. They should formulate questions, interpret information, and form conclusions.

Level 3 Performance

Responses should include a correctly labelled periodic graph with two cycles. Students should clearly explain the relations between period and perimeter and between minimum perimeter and partial perimeter.

Questions with a Functions and Relations:

Focus:

1–6, 8, 9, 11–13, 15, 17, 19–21, 23–25

Questions with a Functions Focus:

1–6, 8, 9, 11–13, 15, 17, 19–21, 23

The Chapter Problem—How Much Daylight?

By the end of this section, students should be able to draw the graph of average monthly hours of daylight versus time (in months). They should be able to identify the typical shapes of the sine and cosine graphs in their graphs.

The Chapter Problem Answers are on p. 5-61 of the *Teacher Resource*.

Challenge 1

Remind students that they should be able to

- recognize that a graph of the average monthly temperature should represent a trigonometric function
- that this graph should be sinusoidal
- determine the properties of the function from its graph

Challenge 2

Remind students that they should be able to

- explain how waves are generated
- explain how the length of a wave is determined
- relate the height and depth of a wave to its maximum and minimum values
- begin to design their wave action pool

The Challenge Answers are on p. 5-68 of the *Teacher Resource*.