

5.8 Solving Linear Trigonometric Equations

Specific Expectations

- TF2.06 solve linear trigonometric equations on the interval $0 \leq x \leq 2\pi$
- TF2.07 demonstrate facility in the use of radian measure in solving equations and in graphing
- TF3.05 write the equation of a sinusoidal function, given its graph and given its properties
- TF4.04 pose and solve problems related to models of sinusoidal functions drawn from a variety of applications, and communicate the solutions with clarity and justification, using appropriate mathematical forms

Evidence of Learning

Students should be able to

- interpolate from a graph to solve a linear trigonometric equation
- use a scientific calculator or a graphing calculator, the CAST rule, or a graph to determine all the solutions of a linear trigonometric equation
- use graphing technology to solve linear trigonometric equations

Text Page 468

Time Required Functions and Relations: 75 min

Time Required Functions: 150 min

Materials and Equipment

- scientific calculator, graphing calculator

Overview

- Solve simple trigonometric equations by interpolating directly from the graph of, for example, $2 = 3 \sin 2(\theta - 30^\circ) + 3$.
- Solve trigonometric equations when $k = 1$ or $k \neq 1$.
- Use a graphing calculator to solve a trigonometric equation.
- Solve a trigonometric equation for a given domain expressed in either degrees or real numbers.

Linear Trigonometric Equations

For Part 1, students graph a sinusoidal function using a graphing calculator and then locate specific points that satisfy certain conditions. This exploration is very visual and should help students actually see the number of possible solutions.

In Part 2, students see how to solve trigonometric equations by graphing, by using algebra, and by using the graphing calculator in different ways.

Management Options

Groups – Arrange students into pairs with at least one graphing calculator per pair. Increase the size of the groups, if needed, so that each group has a calculator. Have students work through Part 1 and examples 1 and 2 in Part 2. For Example 3, demonstrate how the CAST rule, related acute angle, and period are used to determine the angle in radians. Demonstrate Example 4 as well. Have students read through Example 5 and remind them to take notes. Consolidate the example at the board or overhead. Have each group work through examples 6 and 7. Demonstrate solutions using the TI-83 view screen.

Teacher Demonstration – Introduce the meaning of a solution to a linear trigonometric equation on the board, the overhead, or using the TI-83 view screen. Lead students through the *Think, Do, Discuss* in Part 1. Have students read Example 1 and ask some students to go to the board or overhead and solve variations of the question

in this example, for example, $\sin \theta = -0.5$ for $0^\circ \leq \theta \leq 720^\circ$ or $\sin \theta = 0.5$ for $0^\circ \leq \theta \leq 720^\circ$. Lead students through examples 3, 4, and 5. Ask them to do examples 6 and 7 in pairs or individually. Ensure that students find all the solutions for the given domain.

Independent Work – Ask students to follow the instructions in the text on their own. At various points in the lesson, select individuals to summarize the ideas or steps in each part. Students can do this orally or visually on the board, overhead, or the TI-83 view screen.

TECHNOLOGY SUGGESTION

Use the TI-83 view screen throughout this section.

TEACHER INTERVENTION

Review with students the CAST rule and related acute angles. Also, review sketching the graphs of, for example, $y = \sin k\theta$ and $y = \cos k\theta$ over a given domain. The notion of period is also important for finding all possible solutions.

Part 1: Investigating the Number of Solutions

Think, Do, Discuss, text page 468

1. Use a graphing calculator and set the mode to radian.

The mode is set to radians, because the variable t is a measure of time in months, that is, in real numbers.

2. Enter the sales function into **Y1** of the equation editor. Ensure that you enter the equation as shown above.

```

Plot1 Plot2 Plot3
Y1=32.4sin(π/6X)
Y2=53.5
Y3=
Y4=
Y5=
Y6=

```

3. Adjust the window. What must be the minimum and maximum settings for the x- and y-axes so that you will see the first year of the function in the graph?

Set **Xmin** to 0. Set **Xmax** to 12, but change this setting as needed for other steps.

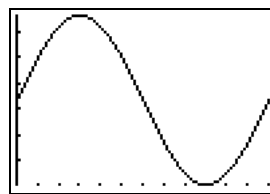
Set **Ymin** to 0 and set **Ymax** to 90.

Pressing **ZOOM** **0** in step 4 will automatically maximize the graph on the screen and reset the window settings.

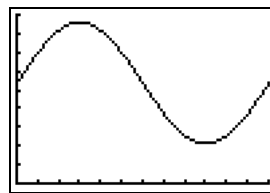
By pressing **GRAPH** instead of zooming, you can see only a portion of the graph.

4. Set the window to display the graph for one year and graph the function by using **ZoomFit**. (Press **ZOOM** **0**.)

Using **ZoomFit**,



Use **GRAPH**.



5. What is the period of this function?

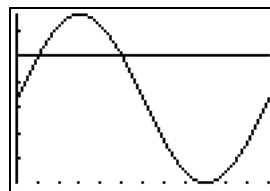
The period is 12 months.

6. In which month are monthly sales at a maximum? at a minimum?

The monthly sales are at a maximum in March ($t = 3$). They are at a minimum in September ($t = 9$).

7. In which month will 70 000 units be sold? Write the trigonometric equation that corresponds to this situation. Enter 70 into **Y2** of the equation editor and graph.

Sales are measured in thousands of units. Then $S(t) = 70$. Graph both equations and note the intersection point or points.



8. How many points of intersection are

there? What are they and what do they mean?

There are two points of intersection. Use **TRACE** to find that $t = 1$ and $t = 5$. For these values of t , the monthly sales are expected to reach 70 000 units in January and May.

9. How many times will the company sell 70 000 units over the next three years? In which months will this occur?

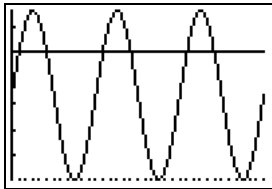
Six times. Assume that the pattern in step 8 is the same each year. Then, for three years, in January and May, the company will sell 70 000 units.

10. Check your predictions by adjusting the window settings for a period of three years and then graph.

```

WINDOW
Xmin=0
Xmax=36
Xscl=1
Ymin=21.118093...
Ymax=85.881906...
Yscl=10
Xres=1

```



11. According to this model, how many times will the company sell 70 000 units over the next ten years?

Twice a year for ten years is 20 times.

12. How many solutions are possible for the equation $70 = 32.4 \sin\left(\frac{\pi}{6}\right)t + 53.5$? Explain.

There are no given restrictions on the domain. For this equation, there are an infinite number of solutions.

Focus on this point and remind students they must use the domain to restrict the number of solutions.

Part 2: Solving Trigonometric Equations

It is important for students to be able to estimate a solution from a graph. Remind students to sketch a

graph each time they solve a trigonometric equation.

Consider drawing a number of different graphs on the board or overhead. Use a horizontal scale in intervals of 90° or $\frac{\pi}{2}$ for

each graph. Select students to locate the solution on each graph. Have a metre stick for the board or a transparent ruler for the overhead at hand.

Alternative Solution for Example 5

Solve $3 \sin(2\theta) + 2 = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

Solution

Rearrange the equation to solve for $\sin(2\theta)$.

$$3 \sin(2\theta) + 2 = 1$$

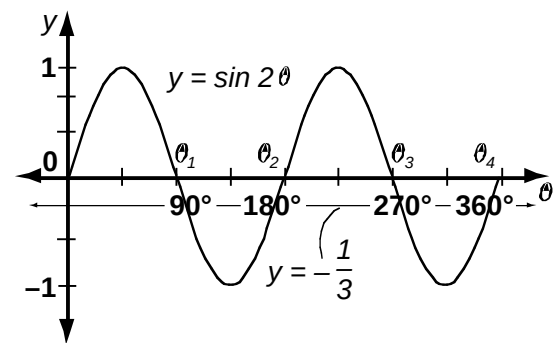
$$\sin(2\theta) = -\frac{1}{3}$$

$$2\theta = \sin^{-1}\left(-\frac{1}{3}\right)$$

$$2\theta \doteq -19.471^\circ$$

$$\theta = -9.735^\circ$$

This angle is not in the given domain. Look at the graph of $y = \sin(2\theta)$.



$$\theta_1 = 90^\circ + 9.735^\circ, \text{ or } 99.735^\circ$$

$$\theta_2 = 180^\circ - 9.735^\circ, \text{ or } 170.265^\circ$$

$$\theta_3 = 270^\circ + 9.735^\circ, \text{ or } 279.735^\circ$$

$$\theta_4 = 360^\circ - 9.735^\circ, \text{ or } 350.265^\circ$$

Consolidate Your Understanding, text page 473

1. Give an example of a linear trigonometric equation and its solutions for a specific domain.

Suggest that students give an example of an equation that is similar to, but not the same as, the equation in Example 5. The example should not be one in this section.

2. *How many solutions are possible for a linear trigonometric equation if the domain is not specified?*

There is an infinite number of solutions.

3. *Which technique described in this section is best for solving the majority of linear trigonometric equations?*

Using a graphing calculator to find the intersection of two relations will give reasonably accurate solutions. As well, solving algebraically by using a scientific or a graphing calculator also gives accurate solutions.

4. *Does a scientific or a graphing calculator always yield all the solutions to a linear trigonometric equation over a given domain? Explain.*

Using an inverse trigonometric function on either calculator gives one solution that may or may not be in the given domain. Use the related acute angles and the period of the function to find all of the solutions in the given domain.

5. *How can you solve any linear trigonometric equation?*

Both solving a system of related equations using a graphing calculator and solving algebraically will produce accurate solutions to any linear trigonometric equation.

TEACHER INTERVENTION

Functions and Relations students should become very fluent in finding algebraic solutions to trigonometric equations.

Additional Example: Functions and Relations

Solve $3 = 2 \cos 3\left(\theta - \frac{\pi}{3}\right) + 4$ for $-\pi \leq \theta \leq 2\pi$ using algebra. Round your answers to one decimal place.

Solution

$$3 = 2 \cos 3\left(\theta - \frac{\pi}{3}\right) + 4$$

$$-1 = 2 \cos 3\left(\theta - \frac{\pi}{3}\right)$$

$$-0.5 = \cos 3\left(\theta - \frac{\pi}{3}\right)$$

$$\cos^{-1}(-0.5) = 3\left(\theta - \frac{\pi}{3}\right)$$

$\cos \theta$ is negative in quadrants II and III.

$\cos^{-1}(-0.5) \doteq 2.09$ in quadrant II and $\pi + (\pi - 2.09) \doteq 4.19$ in quadrant III.

Then $3\left(\theta - \frac{\pi}{3}\right) = 2.09$ and $3\left(\theta - \frac{\pi}{3}\right) = 4.19$.

Solve to get $\theta \doteq 1.74$ and $\theta \doteq 2.44$. The period is $\frac{2\pi}{3}$.

Then the solutions are $1.74 + \left(\frac{2\pi}{3}\right)n$ for

$-2 \leq n \leq 2$ and $2.44 + \left(\frac{2\pi}{3}\right)n$ for $-2 \leq n \leq 1$,

or $\theta = -2.4, -1.7, -0.4, 0.3, 1.7, 2.4, 3.8, 4.5, 5.9$, to one decimal place.

Additional Example: Functions

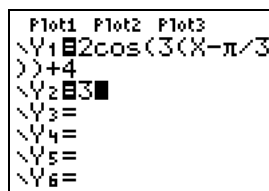
Solve $3 = 2 \cos 3\left(\theta - \frac{\pi}{3}\right) + 4$, $-\pi \leq \theta \leq 2\pi$, by

graphing the intersection of two functions. Round your answers to one decimal place.

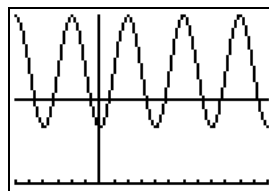
Solution

Use a graphing calculator to graph the two functions.

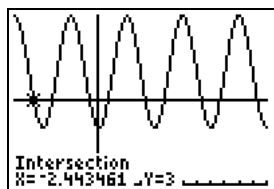
The two functions are



The graph of the two functions is



Press **2nd** **TRACE** **5** and then press **ENTER** three times to locate the first solution.



Repeat these steps until you have found all the solutions for the given domain.

Then $\theta = -2.4, -1.7, -0.4, 0.3, 1.7, 2.4, 3.8, 4.5, 5.9$, to one decimal place.

ASSESSING LEARNING SKILLS

Organization

Observe whether students are prepared. Do they have appropriate materials such as calculators, notebooks, and textbooks? Is the notebook organization easy to follow? Are headings, dates, and highlighting used? Use the Generic Rubric for Organization to assess organization.

Practise, Apply, Solve 5.8

10. Knowledge and Understanding, p. 475

Management Suggestion

Ask students to answer the question independently. After several minutes, have them compare solutions with a partner. Extend the question by replacing x with $2x$ in the equation. Have pairs predict the graph, solve the equation, and evaluate the accuracy of their predictions.

Achievement Chart Connection

Students should demonstrate an understanding that the solutions are restricted by the given domain and show how to locate the solutions on the graph. They must know the algorithms for solving the equations either algebraically or graphically.

Level 3 Performance

Responses should include a graph that shows the solutions. Students should accurately determine the two solutions either algebraically or graphically.

13. Communication, p. 476

Management Suggestion

Have students do the question independently and hand it in for evaluation of form and content.

Achievement Chart Connection

Students should communicate graphically using mathematical symbols and visuals.

Level 3 Performance

Students' graphs should be accurate and neatly labelled, show the line $y = -\frac{1}{\sqrt{2}}$, and show the intersection points and how they relate to the horizontal axis.

14. Thinking, Inquiry, Problem Solving, p. 476

Management Suggestion

Ask students to work in pairs. Decide the city and give students two days for research and collecting data. Ask students to document the Web sites they visit. Collect the finished work a day later.

Achievement Chart Connection

Students should select appropriate technology and resources and organize the data. They should model the data graphically and algebraically, interpret the data to make conclusions, and decide whether their solutions are reasonable.

Level 3 Performance

Students should be able to find a useful sunset/sunrise calculator on the Internet and to prepare a table for the one-year period. They should set an appropriate interval, for example, 15 days, and develop the graphical and algebraic models. After determining on which day(s) the city gets exactly 11 h of daylight, they should assess whether their answers are reasonable and make sense for their community.

19. Application, p. 477

Management Suggestion

Assign this question as independent work. When taking up the answers to the question, have students explain how they decided whether the moving ball was a line drive or a fly ball.

Achievement Chart Connection

Students apply solving a trigonometric equation in the context of hitting a baseball with a bat.

Level 3 Performance

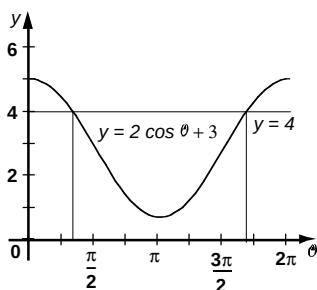
Students should recognize that the horizontal distance a ball travels when it is hit by a bat depends on the impact speeds of the bat and the ball and the angle of impact. They should evaluate the function and solve for a specific angle. As well, they must interpret the answers for a fly ball and a line drive.

23. Check Your Understanding, p. 478

Management Suggestion

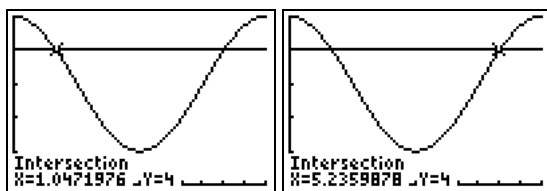
Arrange students into pairs and have them discuss how they would answer (a) and (b). Send various students to the board or overhead to demonstrate (a). Tell their partners they are responsible for assisting. Select other students to demonstrate (b), letting partners assist. Then students should verify using a graphing calculator, while a student or the teacher demonstrates using the TI-83 view screen.

- (a) Since $k = 1$, you can predict that a horizontal line will intersect the curve at two points. Then there are two possible solutions.
- (b) Graph $y = 2 \cos \theta + 3$ and $y = 4$ for $0 \leq \theta \leq 2\pi$, on the same axes.



Then $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(c)



$\frac{\pi}{3} \doteq 1.05$ and $\frac{5\pi}{3} \doteq 5.24$

ASSESSMENT SUGGESTION

Communication

Let students demonstrate their calculator skills using the TI-83 view screen. Look for students' ability to use the appropriate mathematical vocabulary and to explain clearly. Use the last two rows of the Generic Rubric for Communication.

Questions with a Functions and Relations: Focus:

3–5, 6–9 (every other one), 10–19, 21, 23–25

Questions with a Functions Focus:

1–14, 16–19, 21, 23

The Chapter Problem—How Much Daylight?

By the end of this section, students should be able to solve the equation

$h(t) = 7 \sin \frac{\pi}{6}(t-2) + 12.5$ when $h(t) = 13$ by using algebra and a graph.

The Chapter Problem Answers are on p. 5-61 of the *Teacher Resource*.

Challenge 1

Remind students that they should be able to

- describe in detail the mathematical model and how it fits the data
- use the model to determine average monthly temperature
- use the model to determine the month for any given average monthly temperature

Challenge 2

Remind students that they should be able to

- construct a physical model that demonstrates how waves move through a wave action pool

The Challenge Answers are on pp. 5-68 of the *Teacher Resource*.