

6.2 Solving Trigonometry Problems in Two and Three Dimensions

Specific Expectations

The following expectation is common to both the Functions and Relations and the Functions courses.

- TF1.02 solve problems in two dimensions and three dimensions involving right triangles and oblique triangles, using the primary trigonometric ratios, the cosine law, and the sine law (including the ambiguous case)

Evidence of Learning

Students should be able to

- interpret information to draw a diagram and assign variables
- distinguish problems in two dimensions from those in three dimensions
- select the appropriate trigonometric ratio or law to solve problems

Text Page 515

Time Required Functions and Relations:
105 min

**Time Required Functions and Relations
or Functions:** 150 min

Materials and Equipment

- scientific (or graphing) calculator
- stiff paper, scissors

Overview

- Review problem solving in two dimensions in both acute and oblique triangles.
- Learn to represent information as horizontal and vertical triangles for problems involving three dimensions.

Solving Trigonometry Problems in Two and Three Dimensions

This unit is particularly important for students who will attend technology and apprenticeship programs. Many students will need concrete materials to help visualize the situations presented, since this is their first encounter with three dimensions in the mathematics class.

MANAGEMENT SUGGESTION

Have paper, glue, and scissors or other materials available for students to model the problems.

Management Options

Groups – Because this section introduces students to problems involving three dimensions, students with weak visualization skills should be partnered with a student who has stronger skills. Strong academic skills do not necessarily indicate strong visualization skills.

Teacher Demonstration – Not recommended.

Independent Work – Not recommended.

Part 1: Problems in Two Dimensions

Think, Do, Discuss, p. 515

1. *When solving any type of trigonometry problem, what will help you visualize the problem?*

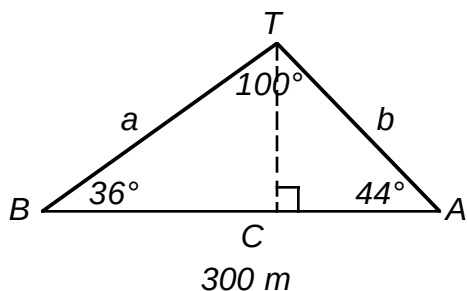
A well-labelled diagram will help you visualize the problem. A good diagram helps you check the reasonableness of the answer and also notice any extra or unneeded information.

TEACHER INTERVENTION

Remind students that the sine law or the cosine law should only be applied to one triangle at a time. Identifying triangles may be confusing when they appear to overlap.

Also, tell students to check the number of planes. There are two dimensions if the information indicates one plane. Students should be familiar with this type of situation, but working with oblique triangles is a new topic.

2. Draw a well-labelled sketch of the situation in the problem. Label all vertices and sides, and label all the given information.



3. For which kinds of triangles can you use the primary trigonometric ratios? Do you need to use these in this problem?

You can use the primary trigonometric ratios for a right triangle. You will need to use right triangles in this problem since the line representing the width of the river is perpendicular to the base line.

4. What information do you need to use the cosine law to solve a problem?

To use the cosine law, you need to know SSS or SAS (contained angle).

5. Do you need to use the cosine law in this problem?

No, it is not necessary to use the cosine law.

6. What do you need to know to use the sine law to solve a problem?

To use the sine law, you need to know ASA or SSA.

7. Do you need to use the sine law in this problem?

Yes, you need to use the sine law.

8. List the steps to follow to solve this problem.

- Calculate the angle at the tree $[180^\circ - (36^\circ + 44^\circ)]$.
- Calculate one of the unknown sides of the obtuse triangle by applying the sine law.
- Draw the altitude from the tree to the riverbank.
- Use one right triangle and a trigonometric ratio to calculate the altitude.

TEACHER INTERVENTION

Because there are several triangles in this diagram, remind students that it is good practice to preface any calculations with "in $\triangle ABC$ " so that someone reading the solution knows where to look for the triangle.

9. Carry out the steps. How wide is the river to the nearest tenth of a metre?

In $\triangle TAC$,

$$\begin{aligned}\frac{a}{\sin 36^\circ} &= \frac{300}{\sin 100^\circ} \\ a &= \frac{300 \sin 36^\circ}{\sin 100^\circ} \\ a &= \frac{300(0.5877852523)}{0.984807753} \\ a &= 179.0558362\end{aligned}$$

Store this value in the calculator's memory for use later.

In $\triangle TBC$, x is the width of the river.

$$\begin{aligned}\sin 44^\circ &= \frac{x}{a} \\ x &= a \sin 44^\circ \\ x &= 179.0885362(0.6946583705) \\ x &\doteq 124.4\end{aligned}$$

The river is about 124.4 m wide.

10. Will the river always be this wide? What other factors should the surveyor think about when building a bridge that can be used through all seasons? Explain.

Depending on the weather conditions in the area, the river may flood (get wider) or become clogged with ice (get more narrow). The surveyor should also think about how people may use the river, for example, for sports or for shipping.

Part 2: Problems in Three Dimensions

MANAGEMENT SUGGESTION

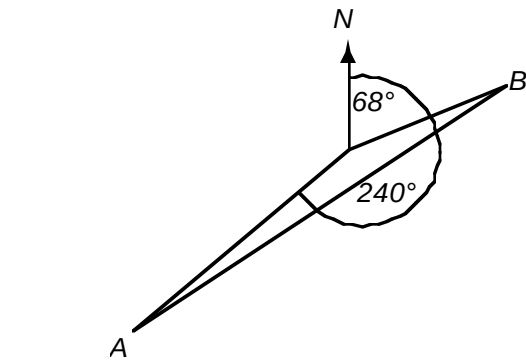
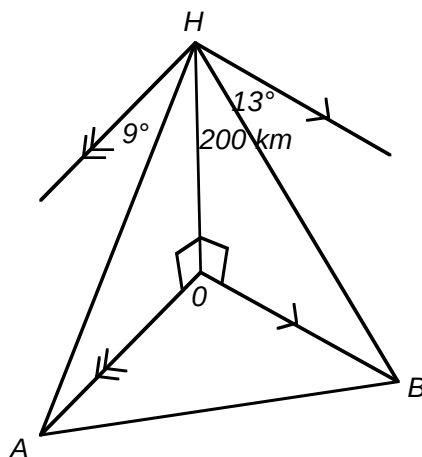
Working with a physical model will help students visualize the problem in three dimensions. Have team members draw two triangles, $\triangle HOA$ and $\triangle HOB$, as shown in the *Think, Do, Discuss*, on stiff paper, cut out the triangles, and stand them up on the desk so that B is on a bearing of 68° from north and A is on a bearing of 240° from north (imagine a compass on the surface of the desk). Alternatively, ask two students to demonstrate by treating the overhead light fixture as the helicopter and standing at the indicated bearings. Students should note that hiker B is closer to O than hiker A since the angle of depression to B is larger.

Think, Do, Discuss, p. 516

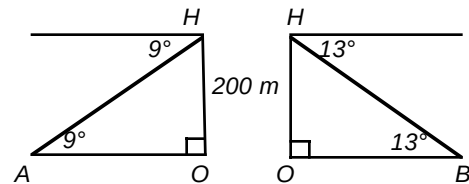
Follow the problem-solving model.

1. Understand the problem.

(a)



(b)

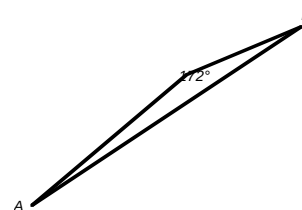


In $\triangle HOA$ and $\triangle HOB$, which are the triangles in the vertical plane,

$HO = 200$ m, which is the altitude of the helicopter

$\angle HAO = 9^\circ$ and $\angle HBO = 13^\circ$, because these angles are *alternate* angles to the angles of depression

In $\triangle OAB$, which is the triangle in the horizontal plane, $\angle AOB = 172^\circ$, because hiker A is at a bearing of 240° , measured from the north, hiker B is at a bearing of 68° , and the angle between the hikers is $240^\circ - 68^\circ = 172^\circ$.



At this point, you cannot know if any of the information is not needed.

(c) and (d) You know one angle. To find the horizontal distance AB , you need, for each triangle, to know either the other two sides or another angle and one side. You could apply the sine law for AAS or SSA, or you could apply the cosine law for SAS.

- (e) Since $\triangle AOB$ shares sides with $\triangle HAO$ and $\triangle HBO$, calculate AO in $\triangle HAO$ and BO in $\triangle HBO$, before you use the cosine law to calculate AB in $\triangle AOB$.
2. *Create a mathematical model of the problem.*
- (a) See the diagrams in question 1 (b).
 (b) See the diagrams in question 1 (b). Use the vertical triangles to calculate the two sides of the horizontal triangle, AO and BO . Use the horizontal triangle to calculate AB , the distance between the hikers.
 (c) Use the tangent ratio for each vertical triangle to calculate AO and BO . Use the cosine law for the horizontal triangle to find AB .
3. *Plan a solution.*
- (a) Use each vertical, right triangle and the tangent ratio to calculate AO and BO .
 (b) See the response to question 1 (b).
 (c) Use the horizontal, obtuse triangle and the cosine law to calculate the distance between the hikers, AB .
 (d) See the response to question 3 (c).

4. *Execute the plan.*

In vertical $\triangle HAO$, $\angle A = 9^\circ$ (alternate angles).

$$\tan 9^\circ = \frac{200}{AO}$$

$$AO = \frac{200}{\tan 9^\circ}$$

$$AO = 1262.750\ 303$$

Store this value in the calculator's memory for later use.

In vertical $\triangle HBO$, $\angle B = 13^\circ$ (alternate angles).

$$\tan 13^\circ = \frac{200}{BO}$$

$$BO = \frac{200}{\tan 13^\circ}$$

$$BO = 866.295\ 174\ 9$$

Store this value in the calculator's memory for later use.

In horizontal $\triangle AOB$,

$$AO = 1262.750\ 303$$

$$BO = 866.295\ 174\ 9$$

$$\begin{aligned}\angle AOB &= 240^\circ - 68^\circ \\ &= 172^\circ\end{aligned}$$

Apply the cosine law.

$$AB^2 = AO^2 + BO^2 - 2(AO)(BO) \cos 172^\circ$$

$$\begin{aligned}AB^2 &= 1262.750\ 303^2 + 866.295\ 174\ 9^2 \\ &\quad - 2(1262.750\ 303)(866.295\ 174\ 9) \\ &\quad \times \cos 172^\circ\end{aligned}$$

$$AB^2 = 1\ 594\ 538.328 + 750\ 467.3301$$

$$- 2\ 187\ 828.989\ (-0.990\ 268\ 086\ 7)$$

$$AB^2 = 4\ 511\ 542.846$$

$$AB = 2124.039\ 276$$

Therefore, the hikers are about 2100 m apart.

MANAGEMENT SUGGESTION

Example 3, page 521, is very similar to the situation in this *Think, Do, Discuss*.

5. *Interpret and evaluate the solution.*

- (a) It is reasonable that $BO < AO$, because the larger angle of depression indicates hiker B is closer to the perpendicular. The distance between the hikers is the longest side of the triangle, because it is opposite an obtuse angle.
- (b) Since the measurements are given to the nearest hundred, the final answer should be rounded to the same degree of accuracy (although the hikers would probably appreciate more accuracy).

6. *Generalize your results.*

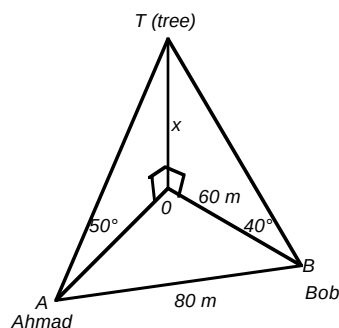
- (a) Yes, you can use this method, provided you have all the information.
- (b) Yes, you can use this method, provided you have all the information.

Additional Example: Functions and Relations

Ahmad and Bob want to calculate the height of a rare tree. They measure the angles of elevation to the top of the tree. The angle of elevation from Bob, who is standing 60 m from the base of the tree, to the top of the tree is 40° . The angle of elevation from Ahmad, who is standing 80 m from Bob, is 50° . Ahmad, Bob, and the tree are not in the same plane.

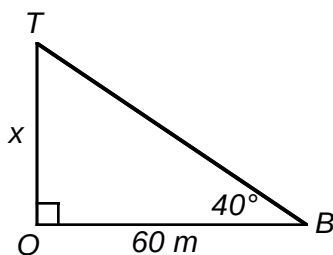
- Calculate the height of the tree.
- Calculate the angle between Ahmad and Bob.
- Suppose Ahmad, Bob, and the tree are in the same plane. Draw a new diagram. Describe, but do not calculate, the following:
 - how to determine the height of the tower
 - how to determine the distance between Ahmad and Bob
- What do you assume?

Solution



- Let the height of the tree be x .

In vertical $\triangle TOB$, $\tan 40^\circ = \frac{x}{60}$.

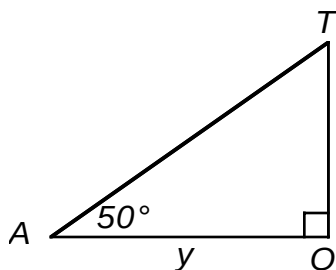


$$x = 50.345\,977\,87$$

(Store this value in the calculator's memory for later use.)

Therefore, the tree is about 50 m tall.

- The distance between Ahmad and the base of the tree is y .



In vertical $\triangle TOA$,

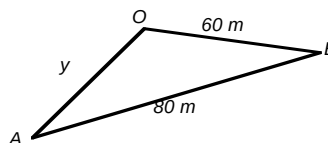
$$\tan 50^\circ = \frac{50.345\,977\,87}{y}$$

$$y = \frac{50.345\,977\,87}{\tan 50^\circ}$$

$$y \doteq 42.245\,291\,46$$

Store this value in the calculator's memory for later use.

Apply the cosine law to calculate the angle between Ahmad and Bob.



$$AB^2 = y^2 + OB^2 - 2y(OB) \cos \angle AOB$$

$$80^2 = 42.245\,291\,46^2 + 60^2 - 2(42.245\,291\,46)(60) \times \cos \angle AOB$$

$$6400 = 1784.664\,651 + 3600 - 5069.434\,975 \cos \angle AOB$$

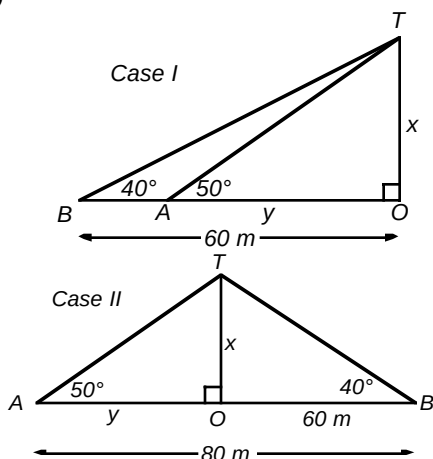
$$1015.335\,349 = -5069.434\,975 \cos \angle AOB$$

$$\cos \angle AOB = -0.200\,285\,703\,2$$

$$\cos \angle AOB = 101.553\,666\,7$$

Therefore, the angle between Ahmad and Bob is about 102° .

(c)



Case I: Ahmad and Bob are on the same side of the tree.

Case II: Ahmad and Bob are on opposite sides of the tree.

In both cases, the problem is in two dimensions, and use right triangles to solve for the height of the tree and for the distance of Ahmad from the base of the tree, just as in (a) and (b).

In case I, where Ahmad and Bob are on the same side of the tree, the distance between them is the difference between their distances from the base of the tree. Bob is farther away from the tree since the angle of depression from Bob to the tree is smaller.

In case II, where Ahmad and Bob are on opposite sides of the tree, the distance between them is the sum of their distances from the base of the tree.

- (d) It was assumed that the tree has no diameter and that Ahmad and Bob are standing on level ground.

Additional Example: Functions

Ahmad and Bob want to calculate the height of a rare tree. They measure the angles of elevation to the top of the tree. The angle of elevation from Bob, who is standing 60 m from the base of the tree, to the top of the tree is 40° . The angle of elevation from Ahmad, who is standing 80 m from Bob, is 50° . Ahmad, Bob, and the tree are not in the same plane.

(a) Calculate the height of the tree.

(b) Calculate the angle between Ahmad and Bob.

Solution

See the answers to (a) and (b) for the Additional Example: Functions and Relations.

Practise, Apply, Solve 6.2

Questions with a Functions and Relations Focus:

1–11, 13, 14, 16–20

Questions with a Functions Focus:

1–9, 11, 12, 14, 16–19

8. Communication, p. 523

Achievement Chart Connection

Thinking of a real-life problem will help students see the relevance of classroom concepts and retain ideas. In explaining what must be done to solve the problem, students will need to follow the steps of a problem-solving model, as outlined in this section.

Management Suggestion

Partners could exchange the problems for this question and attempt to draw a diagram. Then they could compare results and discuss. After listening, request or suggest clarification. You might collect some of the responses to this question for assessment.

Level 3 Performance

Students should

- understand the application of horizontal and vertical triangles that overlap
- include clearly labelled diagrams
- explain the solution clearly and with justification

9. Knowledge and Understanding, p. 523

Achievement Chart Connection

Students are expected to apply the sine law to calculate two sides in an obtuse triangle. The case is not ambiguous.

Level 3 Performance

Students should provide a reasonable sketch and state and apply the sine law accurately.

14. Application, p. 524**Achievement Chart Connection**

Students should demonstrate understanding of the concept of bearing in an unfamiliar setting.

Management Suggestion

Ask that students draw diagrams independently. After a few minutes, have partners discuss their diagrams. Then develop a diagram with the whole class. The teacher or a pair of students can draw this diagram.

Level 3 Performance

Students should

- draw a diagram with a compass indicator with N, S, E, and W at both A and B
- recognize that B 's south is parallel to A 's north before accurately calculating the angles
- apply the sine law and calculate accurately

15. Remind students that there are two cases to consider: one where the car and truck travel on roads that are at a 48° angle to each other and the other where the roads are at a 132° angle to each other. Students should not need a reminder for question 17.

16. Have students write or tape an explanation for a classmate who was absent for the lesson. Then a partner can try to follow the directions. By experiencing the misunderstandings that may ensue, students learn to appreciate clear expression.

18. Thinking, Inquiry, Problem Solving, p. 524**Achievement Chart Connection**

Students must organize several steps to answer this question and recall the mathematical formulas for the surface area and volume of a pyramid.

Management Suggestion

Students should refer to Example 2, page 519. You might review calculating angles in a regular polygon. Guide the first student(s) who has a question. Then this student could help another student who has a similar question, and so on. Students clarify and consolidate their learning when they teach and explain to peers. The trigonometry is routine once the problem is dissected into manageable parts.

Level 3 Performance

Students should

- provide several diagrams from different perspectives
- isolate figures from the whole
- calculate angles, areas, and volume
- present an organized solution

ASSESSMENT SUGGESTION

To encourage students to persevere and to review and revise their solutions, ask students to hand in their earlier rough drafts of the solution for assessment.

19. Check Your Understanding, p. 525**Management Suggestion**

Suggest students record their answers to this question in their journals. Questions of this type encourage students to look at the big picture rather than repeating memorized solutions. Teams should share their responses to reinforce understanding. The responses to this question provide an opportunity for summative assessment. The response, as a narrative, a summary chart, or a combination of both, should not simply duplicate text notes or teacher notes.

20. As a whole class, discuss how to calculate all the necessary angles before applying trigonometric ratios. Then plan a solution with the class.

The Chapter Problem—What Time Is It?

The questions in this section are similar to the questions in section 6.1, except they refer to radian measure. Students in the Functions and Relations course may use a more theoretical approach, while students in the Functions course may find a more pragmatic solution to question CP6.

The Chapter Problem Answers are on p. 6-36 of the *Teacher Resource*.

Challenge 1

For this challenge, students may use diagrams or a concrete model to represent the pyramid, and both models are valid, but students in the Functions course will use concrete models in the future. Some students will excel at modelling, while others may not. Observe students' communication skills while pairs or teams tackle this challenge. If some students need help getting started, suggest they use a cube where each side is 2 units long, since a side length of 1 unit will involve fractions. Students in the Functions and Relations course should represent the side length of the cube with x . Refer students to Example 2.

Remind students that they should be able to complete the challenge.

Challenge 2

Students have not been taught the required skills for this challenge. They will learn about special triangles in section 6.3.

The Challenge Answers are on p. 6-45 of the *Teacher Resource*.