

## 6.5 Skill Builder: Trigonometric Identities

### Specific Expectations

The following expectation is common to both the Functions and Relations and the Functions courses.

- **TF2.05** prove simple identities, using the Pythagorean identity,  $\sin^2 x + \cos^2 x = 1$ , and the quotient relation,  $\tan x = \frac{\sin x}{\cos x}$

### Evidence of Learning

*Students should be able to*

- algebraically prove trigonometric identities by substituting the quotient and Pythagorean identities and then simplifying

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**Time Required Functions and Relations or Functions:** 150 min

### Materials and Equipment

- spreadsheet software

### Overview

- Define an identity.
- Prove the quotient identity.
- Prove a Pythagorean identity.
- Illustrate strategies for proving an identity.

### Trigonometric Identities

Proving identities requires practice and students will need support to avoid frustration. Students' abilities to master this new skill will vary, so allow time for those students who will need it. You might pair a student who adapts quickly with a student who needs support.

### Management Options

**Teacher Demonstration** – This method is best for this lesson.

**Independent Work** – Students could practise reading a math text, supported by either the teacher or a partner. You might assist the first student(s) who has a question and then that student can assist the next student who has the same question, and so on.

### MANAGEMENT SUGGESTION

Students should memorize one form of the Pythagorean identity. Offer an inexpensive prize to the team that states the identity accurately.

### Additional Example: Functions and Relations

Prove the quotient identity by

graphing  $y = \frac{\sin \theta}{\cos \theta}$  for certain points of

each function such that  $f(\theta) = \sin \theta$ ,

$g(\theta) = \cos \theta$ , and  $h(\theta) = \frac{f(\theta)}{g(\theta)}$ .

### MANAGEMENT SUGGESTION

Remind students that the value of a fraction is 0 if the numerator is 0; division by 0 is undefined, that is, the denominator is 0; the value of the fraction is 1 if the numerator equals the denominator; and the value of the fraction is  $-1$  if the numerator equals the negative of the denominator. Using this method, students can easily graph a reciprocal trigonometric relation.

### Solution

(The values for each trigonometric function are rounded to three decimal places.)

For  $\theta = 0$ ,  $f(\theta) = 0$ , so  $h(\theta) = 0$ .

For  $\theta = \frac{\pi}{4}$ ,  $f(\theta) = g(\theta)$ , so  $h(\theta) = 1$ .

For  $\theta = \frac{\pi}{2}$ ,  $g(\theta) = 0$ , so  $h(\theta)$  is undefined.

Draw an asymptote.

For  $\theta = \frac{3\pi}{4}$ ,  $f(\theta) = -g(\theta)$ , so  $h(\theta) = -1$ .

For  $\theta = \pi$ ,  $f(\theta) = 0$ , so  $h(\theta) = 0$ .

For  $\theta = \frac{5\pi}{4}$ ,  $f(\theta) = g(\theta)$ , so  $h(\theta) = 1$ .

For  $\theta = \frac{3\pi}{2}$ ,  $g(\theta) = 0$ , so  $h(\theta)$  is undefined. Draw an asymptote.

For  $\theta = \frac{7\pi}{4}$ ,  $f(\theta) = -g(\theta)$ , so  $h(\theta) = -1$ .

For  $\theta = 2\pi$ ,  $f(\theta) = 0$ , so  $h(\theta) = 0$ .

Therefore, the graph of the quotient identity is the same as graph of the tangent function.

### Additional Example: Functions

Prove the quotient identity by creating a spreadsheet or a table for  $y = \frac{\sin x}{\cos x}$ ,

where  $x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \dots$ . Explain the values in the

spreadsheet for  $y = \frac{\sin x}{\cos x}$  and  $\tan x$  if  $x = \frac{\pi}{2}$ .

### Solution

The following spreadsheet shows the formulas.

Enter **=B1+PI()/6** in cell C1. PI()/6 represents  $\frac{\pi}{6}$ .

Then copy the formulas in cells B2 to B5 and C1. The spreadsheet calculates the sine, the cosine, or the tangent of a number, where the number is an angle in radians, and then you can apply the degree function of the spreadsheet to convert radians to degrees.

|   | A                     | B        |
|---|-----------------------|----------|
| 1 | x                     | 0        |
| 2 | $y = \sin x$          | =sin(B1) |
| 3 | $y = \cos x$          | =cos(B1) |
| 4 | $y = \sin x / \cos x$ | =B2/B3   |
| 5 | $y = \tan x$          | =tan(B1) |

This spreadsheet shows the results.

|   | A                     | B | C      |
|---|-----------------------|---|--------|
| 1 | x                     | 0 | 0.5236 |
| 2 | $y = \sin x$          | 0 | 0.5000 |
| 3 | $y = \cos x$          | 1 | 0.8660 |
| 4 | $y = \sin x / \cos x$ | 0 | 0.5774 |
| 5 | $y = \tan x$          | 0 | 0.5774 |

|     | D      | E      |
|-----|--------|--------|
| ... | 1.0472 | 1.5708 |
| ... | 0.8660 | 1.0000 |
| ... | 0.5000 | 0.0000 |
| ... | 1.7321 | #####  |
| ... | 1.7321 | #####  |

Since the values in row 4 and 5 are

identical, then  $\frac{\sin x}{\cos x} = \tan x$ .

Division by 0 is undefined, so the calculator cannot calculate the values of  $\frac{\sin x}{\cos x}$  and

$\tan x$  for  $x = \frac{\pi}{2}$ .

### MANAGEMENT SUGGESTION

Students who will take business or technology courses will need to be familiar with spreadsheets.

### TEACHER INTERVENTION

#### Example 1

Remind students to always include the angle when writing, or speaking about, a trigonometric function. "tan cos" has no meaning, but  $\tan \theta$  does have meaning.

### TEACHER INTERVENTION

Point out that, in section 6.4, students solved for the value of the variable. The solution for proving an identity is found when both sides are equal to the same expression.

## Practise, Apply, Solve 6.5

### Questions with a Functions and Relations

#### Focus:

1–7, 9, 11–16

### Questions with a Functions Focus:

1–7, 9, 11–13

### Management Suggestion

Students should provide complete solutions and follow proper format when proving identities. Assign one or two parts from questions 1 to 5 for students to solve in class; provide solutions. Near the end of this period, randomly divide questions 6 to 13 among six different teams and tell students to prepare and present solutions for the next class. At the beginning of the next class, allow a short time for each team to polish their presentations. Encourage teams to exchange information, and have the solutions available.

A particular identity can be solved in several different ways, and some solutions will be more efficient than others, so grade solutions accordingly.

#### TEACHER INTERVENTION

When students experience difficulty in proving identities, remind them to persevere, to try another approach, and to look for similarities between the two sides of an equation. It is tempting to skip steps when proving identities, because students know that the two sides are equal in an equation. However, tell students to include all steps in their proofs to avoid making errors.

- To help some students factor these expressions, suggest that they replace  $\sin \theta$  with  $x$ . For example, in question 3 (f),  $\sin^2 \theta - 2 \sin \theta + 1$  becomes  $x^2 - 2x + 1$ , which can be factored as  $(x - 1)^2$ . Then replace  $x$  with  $\sin \theta$  to get  $(\sin \theta - 1)^2$ . Students should use this method to get started, but not to rely on it.
- Alternatively, students may first express the left side of the equation in terms of  $\sin \alpha$ , as the question suggests, then find the common denominator of the terms on the right side and subtract the terms, which will make the right side equal to the left side.

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### Management Suggestion

Partners could exchange answers, discuss differences, and ensure that their solutions are complete. Because many solutions are possible, clarity is critical. Seeing other solutions broadens students' perspectives.

| L.S.                                                             | R.S. |
|------------------------------------------------------------------|------|
| $\sin^2 x \left( 1 + \frac{1}{\tan^2 x} \right)$                 | 1    |
| $= \sin^2 x \left( 1 + \frac{\cos^2 x}{\sin^2 x} \right)$        |      |
| $= \sin^2 x \left( \frac{\sin^2 x + \cos^2 x}{\sin^2 x} \right)$ |      |
| $= \sin^2 x \left( \frac{1}{\sin^2 x} \right)$                   |      |
| $= 1$                                                            |      |

L.S. = R.S., therefore,  $\sin^2 x \left( 1 + \frac{1}{\tan^2 x} \right) = 1$   
for all values of  $x$ .

(See the *Solutions Manual* for another approach.)

This problem is comprehensive, without being complex. The solution involves each step outlined in the *Key Ideas*, as follows:

- Separate the two sides of the expression.
- Simplify the more complicated side until it is identical to the other side.
- Express all tangent functions in terms of the sine function or the cosine function.
- Look for expressions to which you can apply the Pythagorean identity.
- Where necessary, find a common denominator.

### Challenge 2

Remind students that they should be able to complete the entire challenge.

The Challenge Answers are on p. 6-45 of the *Teacher Resource*.