

6.1 Extending Trigonometry Skills with Oblique Triangles

Specific Expectations

The following expectation applies to both the Functions and Relations course and the Functions course.

- TF1.02 solve problems in two dimensions and three dimensions involving right triangles and oblique triangles, using the primary trigonometric ratios, the cosine law, and the sine law (including the ambiguous case)

Evidence of Learning

Students should be able to

- solve obtuse triangles using the cosine law and the sine law
- identify the ambiguous case when using the sine law

Text Page 499

Time Required Functions and Relations:

105 min

Time Required Functions: 150 min

Materials and Equipment

- compass, protractor, ruler
- overhead protractor
- computers with dynamic geometry software
- scientific calculator

Overview

- Define oblique triangle.
- Investigate and prove the cosine law for obtuse triangles.
- Investigate and prove the sine law for obtuse triangles.
- Develop a model for using the sine law in the ambiguous case.

The Trigonometry of Oblique Triangles

Students in the Functions and Relations course must consider the nature of proof in this section. Note the difference between the empirical methods that students will use for *Think, Do, Discuss* questions 1 to 6, Part 1, and the deductive method in question 7. Since students must practise the skills of reading a math text, they should carefully consider each line in question 7 in each part.

Students in the Functions course who may take business courses should record information in a spreadsheet for *Think, Do, Discuss* question 3 of Part 3. Other students, who may take an apprenticeship, should use dynamic geometry software to construct triangles, although they may prefer to hand draw the triangles.

Management Options

Part 1 and Part 2

Groups – Teams could work through each *Think, Do, Discuss* question, except question 7 in each part. For this question, lead a whole-class discussion.

Teacher Demonstration – Lead a whole-class discussion about question 7 of each part.

Independent Work – Independent study is appropriate for students who demonstrate responsibility for their own learning.

Part 3

Groups – If students have access to computers with dynamic geometry software and students are very comfortable with the software, they can work in pairs.

Teacher Demonstration – Demonstrate constructing triangles on large sheets of paper or on the board.

Independent Work – Independent study is appropriate for students who demonstrate responsibility for their own learning.

TEACHER INTERVENTION

Comparing a to $b \sin A$ (see the *Key Ideas*) is a skill worth committing to memory. Suggest that students first determine if the SSA case applies in a situation.

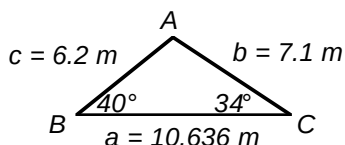
ASSESSMENT SUGGESTION

For students who are familiar with spreadsheet software, suggest designing a spreadsheet to calculate $b \sin A$, to compare the value of $b \sin A$ to a and b , and then to state the number of possible triangles for the SSA case.

Part 1: Investigating the Cosine Law and Obtuse Triangles

Think, Do, Discuss, p. 499

1. Sketch the roof. Starting with the obtuse angle, label the vertices A , B , and C in a counterclockwise direction. Label the corresponding sides of the triangle.



2. Show that the cosine law is true for each acute angle in the diagram.

TEACHER INTERVENTION

Alternatively, calculate each angle, not each side, using the cosine law, which is done in question 5.

$$\angle B = 40^\circ, \angle C = 34^\circ, a = 10.636 \text{ m}, \\ b = 7.1 \text{ m}, \text{ and } c = 6.2 \text{ m}$$

Show that the cosine law applies to $\angle B$.

$$b^2 = a^2 + c^2 - 2ac \cos B \\ b^2 = 10.636^2 + 6.2^2 - 2(10.636)(6.2) \cos 40^\circ \\ b = 7.108 \ 702 \ 565$$

Since b is about 7.1, the cosine law is true for $\angle B$.

Show that the cosine law applies to $\angle C$.

$$c^2 = a^2 + b^2 - 2ab \cos C \\ c^2 = 10.636^2 + 7.1^2 \\ \quad - 2(10.636)(7.1) \cos 34^\circ \\ c = 6.190 \ 634 \ 586$$

Since c is about 6.2, the cosine law is true for $\angle C$.

3. Determine $\cos 106^\circ$. How does this cosine differ from the cosines of the other two angles in the triangle?

$\cos 106^\circ = -0.275 \ 637 \ 355 \ 8$. This cosine is negative, but the cosines of the two acute angles are positive.

4. How does the cosine of any obtuse angle differ from the cosine of any acute angle? Does this prevent the cosine law from being true for the obtuse angle?

The cosine of any obtuse angle is negative, but the cosine of any acute angle is positive. The cosine law appears to be true for the obtuse angle.

5. Show that the cosine law is true for the obtuse angle.

$$a = 10.636 \text{ m}, b = 7.1 \text{ m}, c = 6.2 \text{ m}, \\ \angle A = 106^\circ$$

Show that the cosine law is true for $\angle A$.

$$a^2 = b^2 + c^2 - 2bc \cos A \\ 10.636^2 = 7.1^2 + 6.2^2 - 2(7.1)(6.2) \cos A$$

Solve to get $\cos A = -0.275 \ 721 \ 217 \ 6$, so $\angle A \doteq 106^\circ$.

Then the cosine law is true for $\angle A$.

6. Based on the results from step 5, can you say that the cosine law will work in all obtuse triangles? Explain.

From this one example, it seems that the cosine law will work for an obtuse triangle, but the general case should be proved.

7. Examine this proof of the cosine law for any obtuse triangle, where $\angle B$, or $\angle ABC$, is obtuse.

See page 500.

MANAGEMENT SUGGESTION

Students should be able to actively follow proofs, even though they might not be able to create them. Reading a proof is not the same as reading a novel or an instruction manual. Suggest students ask themselves the following questions as they read the proof:

“Does this statement always work?”

“Where did this statement come from?”

“Could this statement be more concise?”

“What is assumed?”

Read the proof together with the whole class or direct individuals or teams to read the proof. Periodically question individuals or teams to check understanding.

Once the class has read question 7, have one pair from one team prove $a^2 = b^2 + c^2 - 2bc \cos A$, and have another pair from another team prove $c^2 = a^2 + b^2 - 2ab \cos C$ so that students might discuss questions with a teammate.

8. In the proof above,

- (a) What does h represent in $\triangle ABC$, $\triangle ADC$, and $\triangle BDC$?

h represents the altitude in all three triangles.

- (b) Explain why the two expressions for h^2 can be set equal to each other in step iii.

h represents the same line segment, CD , so the expressions representing h^2 are the same.

- (c) Explain why $\cos(180^\circ - \theta) = -\cos \theta$ in step v.

The trigonometric ratios of two related angles have the same absolute value, but different signs. The cosine of acute angle θ is positive, but the cosine of the related, obtuse angle $180^\circ - \theta$ is negative. Therefore,
 $\cos(180^\circ - \theta) = -\cos \theta$.

9. What is the least information you need to know about a triangle to use the cosine law to calculate

- (a) an unknown side?

To calculate an unknown side using the cosine law, you must know both other sides and the angle contained by those two sides (SAS).

- (b) an unknown angle?

To calculate an unknown angle using the cosine law, you must know all three sides (SSS).

EXTENSION

Discuss how the cosine of $\angle A$ affects the magnitude of side a if

- i. $\angle A = 0^\circ$ ii. $\angle A$ is acute iii. $\angle A = 90^\circ$
 iv. $\angle A$ is obtuse v. $\angle A \geq 180^\circ$

Students in the Functions course may refer to specific angles; encourage students in the Functions and Relations course to analyze on a more abstract basis, where possible.

Part 2: Investigating the Sine Law and Obtuse Triangles

Refer to the diagram of the roof cross section from Part 1.

Think, Do, Discuss, p. 502

1. Show that the sine law is true for each acute angle in the diagram.

$a = 10.636$ m, $b = 7.1$ m, $c = 6.2$ m,
 $\angle B = 40^\circ$, $\angle C = 34^\circ$

Substitute the values to show that

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

L.S.	R.S.
$\frac{b}{\sin B}$	$\frac{c}{\sin C}$
$= \frac{7.1}{\sin 40^\circ}$	$= \frac{6.2}{\sin 34^\circ}$
$\doteq 11$	$\doteq 11$

Therefore, L.S. = R.S. and $\frac{b}{\sin B} = \frac{c}{\sin C}$.

2. Determine $\sin 106^\circ$. Other than value, does this sine differ from the sines of the other two angles in the triangle in any way?

$\sin 106^\circ = 0.961\ 261\ 695\ 9$. No, the sines of both acute and obtuse angles are positive.

3. Determine $\sin 74^\circ$. What do you notice?

$\sin 74^\circ = 0.961\ 261\ 695\ 9$. The values for $\sin 106^\circ$ and $\sin 74^\circ$ are the same.

4. When using the sine law to calculate the length of a side, is the calculation affected at all if the angle is obtuse, rather than acute or right?

Since the sines of all these angles are positive, there is no effect on the calculation.

5. Show that the sine law is true for the obtuse angle in the diagram.

$$a = 10.636\text{ m}, b = 7.1\text{ m}, \angle A = 106^\circ, \angle B = 40^\circ$$

Substitute the values to show that

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

L.S.	R.S.
$\frac{a}{\sin A} = \frac{10.636}{\sin 106^\circ}$	$\frac{b}{\sin B} = \frac{7.1}{\sin 40^\circ}$
$\doteq 11$	$\doteq 11$

Therefore, L.S. = R.S. and $\frac{a}{\sin A} = \frac{b}{\sin B}$.

6. Based on your results in step 5, can you say that the sine law will work in all obtuse triangles? Explain.

From this one example, it seems that the sine law will work in one obtuse triangle, but the general case for all triangles should be proved.

7. Examine this proof of the sine law for any obtuse triangle, where $\angle B$, or $\angle ABC$, is obtuse.

See page 502.

8. In the proof in step 7, explain why $\sin (180^\circ - \theta) = \sin \theta$.

The trigonometric ratios of two related angles have the same value and sign.

The sine of acute angle θ is positive and so is the sine of the related, obtuse angle $180^\circ - \theta$. Therefore,
 $\sin (180^\circ - \theta) = \sin \theta$.

9. What is the least information you need to know about the triangle to use the sine law to calculate

(a) an unknown side?

To calculate an unknown side using the sine law, you must know any side and two angles (AAS) or (ASA).

(b) an unknown angle?

To calculate an unknown angle using the sine law, you must know two sides and an angle opposite one of those sides (SSA).

Part 3: A Closer Look at the Sine Law—The Ambiguous Case

TECHNOLOGY SUGGESTION

Students in the Functions course who may take an apprenticeship should use dynamic geometry software. The steps for constructing triangles with *The Geometer's Sketchpad* appear in *Mathematics 10*, pages 450 to 454, and there is more information in the corresponding section in the *Teacher Resource* for this text.

Think, Do, Discuss, p. 503

1. Draw a line segment AB at least 15 cm long. Construct line segment AC so that $\angle CAB = 40^\circ$ and $A = b = 10\text{ cm}$.

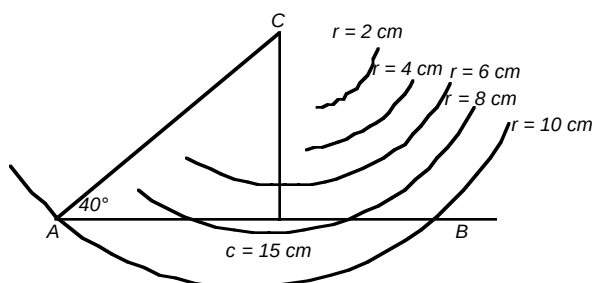
Line segment AB must be at least 15 cm long to facilitate the construction of the triangle. For $\triangle ABC$, $\angle A = 40^\circ$ and $b = 10\text{ cm}$. For the next two steps, a , or r , will change to 2 cm, 4 cm, 6 cm, 8 cm, and 10 cm. See the diagram in question 3. The underlying case is SSA.

2. Construct an arc that originates from point C with a radius of 2 cm. Does the arc intersect AB?

An arc with radius 2 cm does not intersect AB. See the diagram in question 3.

3. Repeat step 2 using the radii from the table. Record the number of triangles that can be drawn with each radius.

Radius, r (cm)	Number of Triangles
2	0
4	0
6	0
8	2
10	1



Note that r and a are the same in this example. Also, $a + b \geq c$, or a triangle cannot be formed.

4. Determine the radius that would allow you to draw exactly one triangle.

For one triangle, r must be perpendicular to AB. See the diagram in question 3. The arc with radius 6 cm is close, but too short.

$$\sin 40^\circ = \frac{r}{10}$$

$$r = 10 \sin 40^\circ$$

$$r = 6.427\ 876\ 097$$

Therefore, a radius of about 6.4 cm would form exactly one triangle.

5. Calculate the value of $b \sin A$. What does this represent in your diagram?

$$b = 10 \text{ cm}, \angle A = 40^\circ$$

$$b \sin A = 10 \sin 40^\circ$$

$$= 6.427\ 876\ 097$$

$b \sin A$ represents the side length required to form one triangle. It is also the altitude of the triangle from C.

6. For which values of r does the arc intersect AB twice?

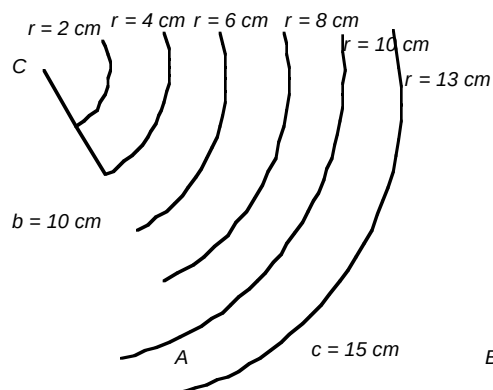
For any values of r such that $b \sin A < r < b$, the arc intersects AB twice.

7. For the radii that allowed you to create two triangles, determine the height of each triangle. What do you notice?

$$\text{The height is } \sin 40^\circ = \frac{r}{10}.$$

$$r > b \sin A$$

8. Repeat steps 1–3 using $\angle A = 120^\circ$. Under what conditions do you get a triangle? do not get a triangle?



$$AB = 15 \text{ cm}, \angle CAB = 120^\circ,$$

$$AC = b = 10 \text{ cm},$$

$$CB = a = 2 \text{ cm}, 4 \text{ cm}, 6 \text{ cm}, 8 \text{ cm}, \text{ and } 10 \text{ cm}$$

Students should see that the position of point B changes and that a triangle may be drawn at the point where the arc intersects AB.

Radius, r (cm)	Number of Triangles
2	0
4	0
6	0
8	0
10	0
13	1

Note that r must be greater than 10 cm to form a triangle. Comparing r to $b \sin A$ (or the altitude) is irrelevant, because the altitude from C is outside the triangle.

TECHNOLOGY SUGGESTION / EXTENSION

Students in the Functions course who may study business could design a spreadsheet such that the values of a , b , and $\angle A$ are input, $b \sin A$ is calculated, a , b , and $b \sin A$ are compared, and the number of possible triangles is output.

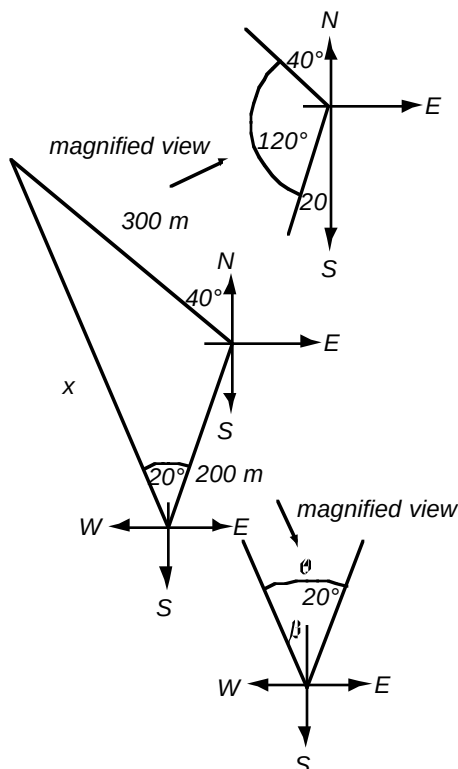
Additional Example: Functions and Relations

A surfsailor at Rondeau Bay sails 200 m on a bearing of 20° , changes direction to $N40^\circ W$, sails 300 m, and stops.

- How far from the original position should a friend sail to meet him?
- In what direction should the friend sail?

Solution

Draw a diagram and include both compass indicators. Bearings are always measured from the north, in a clockwise direction. $N40^\circ W$ means 40° west of north. Calculate the other necessary angles.



Use the cosine law.

$$\begin{aligned}x^2 &= 200^2 + 300^2 - 2(200)(300) \cos 120^\circ \\x^2 &= 190\,000 \\x &= 435.889\,894\,4\end{aligned}$$

Store this answer in the calculator's memory to use later.

Use the sine law.

$$\begin{aligned}\frac{\sin \theta}{300} &= \frac{\sin 120^\circ}{435.889\,894\,4} \\ \sin \theta &= 0.596\,039\,560\,7 \\ \theta &\doteq 37^\circ \\ \beta &= \theta - 20^\circ \\ &= 17^\circ\end{aligned}$$

The friend should sail 436 m at $N17^\circ W$.

MANAGEMENT SUGGESTION

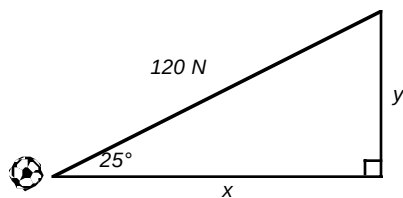
Students who will take the Geometry and Discrete Mathematics course will study problems involving forces and projections. They should also do the next example.

Additional Example: Functions

A soccer player kicks the ball at a 25° angle to the ground with a force of 120 N.

- What is the force that moves the ball forward (the horizontal distance)?
- What is the force that lifts the ball (the vertical distance)?
- What adjustment to the angle should the player make if he wants the ball to move through a greater horizontal distance?
- What adjustment to the angle should the player make if he wants the ball to have a greater vertical lift?

Solution



(a)

$$\frac{x}{\sin 65^\circ} = \frac{y}{\sin 25^\circ} = \frac{120}{\sin 90^\circ}$$

$$\frac{x}{\sin 65^\circ} = \frac{y}{\sin 25^\circ} = \frac{120}{1}$$

$$\frac{x}{\sin 65^\circ} = \frac{120}{1}$$

$$x = 108.756\ 934\ 4$$

$$x \doteq 109$$

A force of about 109 N moves the ball forward through the horizontal distance.

(b)

$$\frac{y}{\sin 25^\circ} = \frac{120}{1}$$

$$y = 50.714\ 191\ 41$$

A force of about 51 N lifts the ball through the vertical distance.

(c) To increase x , the horizontal distance, increase $\sin 65^\circ$, which means increasing the 65° angle and decreasing the 25° angle.

(d) To increase y , the vertical distance, increase $\sin 25^\circ$, which means increasing the 25° angle.

Alternatively, substitute larger or smaller angles and check by applying the sine law. Or, simply draw larger or smaller angles and observe the horizontal and vertical changes.

Practise, Apply, Solve 6.1

Questions with a Functions and Relations Focus:

Focus:

1–10, 12, 14, 16, 17, 19, 21–25

Questions with a Functions Focus:

1–9, 12, 13, 15, 16, 18, 19, 22

9. Knowledge and Understanding, p. 512

Achievement Chart Connection

This question is similar to the question in Example 2, page 507. Students should demonstrate facility with the ambiguous case. They should also list the variables, substitute values, and state a conclusion. They should demonstrate the ability to manipulate a formula using different variables. They need to apply the sine law accurately to calculate angles and side lengths.

Management Suggestion

Remind students that identifying the SSA case should become routine. In class discussion, practise applying the conditions described in the *Key Ideas* to triangles that do *not* have vertices named A , B , and C , for example, $\triangle DEF$.

Level 3 Performance

Students should

- draw a reasonable and correctly labelled sketch
- compare d and $e \sin D$
- state appropriate conclusions
- accurately apply the sine law in both cases

12. Thinking, Inquiry, Problem Solving, p. 513

Achievement Chart Connection

Students must interpret ambiguous information, test possibilities, and draw reasonable conclusions, which must be clearly communicated.

Management Suggestion

Partners should ensure that each partner realizes that there are two possible lots.

Level 3 Performance

Students should

- draw a reasonable diagram of both possible lots
- communicate the relationship between a and $b \sin A$ clearly and interpret the results
- calculate the dimensions of the lot using the sine law
- communicate a clearly organized solution

14. Remind students that they must apply the sine law and the cosine law to one triangle at a time.

16. Communication, p. 513

Achievement Chart Connection

Students must demonstrate understanding of the ambiguous case, which could be communicated either visually, as in the fourth case in the *Key Ideas*, or in their own words, or a combination of both. This question should be an important component of their personal study journal.

Management Suggestion

Students could write their responses as if they were explaining the answer for a classmate who was absent for this lesson.

Level 3 Performance

Students should

- demonstrate understanding of the several conditions involved in the ambiguous case
- communicate this understanding clearly
- use symbols accurately

19. Application, p. 514

Achievement Chart Connection

For an unfamiliar context, students will apply the sine law in an obtuse triangle. Students need to draw a complicated diagram and calculate angles.

Level 3 Performance

Students should

- demonstrate understanding of right and obtuse triangles
- apply the sine law
- interpret vocabulary
- communicate a clearly organized solution

Management Suggestion

Ask that students draw diagrams independently. After a few minutes, have partners discuss their diagrams. Finally, develop a diagram as a whole class, which may be drawn by either the teacher or by a pair of students during the discussion.

22. Check Your Understanding, p. 514

Students must express all the components of the *Key Ideas* in their own words.

Management Suggestion

Have team members exchange results and identify any errors or omissions. This question should be an important component of their journals.

ASSESSING LEARNING SKILLS

Divide the class into teams and distribute the Learning Skills Self-Assessment Checklists. Observe team discussion and collect the completed checklists at the end of the class for assessment.

The Chapter Problem—What Time Is It?

Students will need to draw and solve right triangles for questions CP1 and CP2. (See *Mathematics 10*, section 5.8.) In section 5.6, page 452, students learned how to calculate the period of a sinusoidal function, given its equation.

The Chapter Problem Answers are on p. 6-36 of the *Teacher Resource*.

Challenge 1

Students have had little experience in working in three dimensions in the mathematics class. At this point, they should know the required geometry and trigonometry, but wait until the end of the next section before assigning this challenge. Example 2 in section 6.2, page 519, is particularly informative.

Challenge 2

Students have not been taught the required skills for this challenge. They will learn about special triangles in section 6.3.

The Challenge Answers are on p. 6-45 of the *Teacher Resource*.