

7.2U Revisiting the Circle

Specific Expectations

- LC1.01U construct a geometric model (e.g., a diagram created by hand, a diagram created by using dynamic geometry software) to represent a described locus of points; determine the properties of the geometric model; and use the properties to interpret the locus (e.g., the locus of points equidistant from two fixed points is the right bisector of the line segment joining the two fixed points)
- LC1.02U explain the process used in constructing a geometric model of a described locus
- LC1.03U determine an equation to represent a described locus [e.g., determine the equation of the locus of points equidistant from $(-2, 7)$ and $(5, 4)$]
- LC1.04U construct geometric models to represent the locus definitions of the conics
- LC1.05U determine equations for conics from their locus definitions, by hand for simple particular cases [e.g., determine the equation of the locus of points the sum of whose distances from $(-3, 0)$ and $(3, 0)$ is 10]
- LC2.01U identify the standard forms for the equations of parabolas, circles, ellipses, and hyperbolas having centres at $(0, 0)$ and at (h, k)
- LC2.03U determine the key features (e.g., the centre or the vertex, the focus or foci, the asymptotes, the lengths of the axes) of a conic whose equation is given in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, by hand in simple cases (e.g., $x^2 + 9y^2 - 6x + 36y - 36 = 0$)
- LC2.04U sketch the graph of a conic whose equation is given in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$
- LC2.05U illustrate the conics as intersections of planes with cones, using concrete materials or technology
- LC3.02U pose and solve problems drawn from a variety of applications involving conics, and communicate the solutions with clarity and justification (Sample problem: A parabolic antenna is 320 m wide at a distance of 50 m above its vertex. Determine the distance above the vertex of the focus of the antenna)

Students should be able to

- demonstrate that relationships can be represented in many different ways — geometrically, algebraically, or through a locus definition
- determine the equation of a circle given its centre and the radius
- determine the centre and radius of a circle given its equation in general form
- sketch the graph of a circle given its equation in either standard or general form
- determine the equation of a circle given specific criteria such as three points or the centre and a point on the circle, and so on
- solve problems involving the equation of a circle

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Time Required Functions and Relations:

75 min

Materials and Equipment

- compasses
- graph paper
- *The Geometer's Sketchpad*

Overview

This section reviews what students should know from grade 10 about circles that are centred about the origin. Students will extend the equation of a circle to describe a circle that has centre (h, k) and develop the general form of this equation by completing the square.

MANAGEMENT SUGGESTION

Review graphing linear relations — by creating a table, plotting x - and y -intercepts, or using the slope and y -intercept — before taking up this lesson.

Management Options

Groups – As pairs work on the *Think, Do, Discuss* questions of Part 1, invite students to put their solutions to each question on the board or overhead. Once Part 1 has been discussed, divide students into groups of four. Half of the groups can work on the algebraic solution to the *Think, Do, Discuss* questions in Part 2 and the other half can work on the geometric solution. Students working on the geometric solution can either use *The Geometer's Sketchpad* or graph paper, compasses, and ruler. Compare the answers resulting from the different methods and then summarize the key points.

ASSESSMENT SUGGESTION

Teamwork

If groups explore the *Think, Do, Discuss* Part 2, distribute the Group Participation: Peer-Assessment Rating Scale to students. (See Assessment and Evaluation in the *Teacher Resource*.)

Teacher-Led Demonstration – Lead students through the *Think, Do, Discuss* questions of Part 1. Students should use *The Geometer's Sketchpad* for Part 2. Verify the results through a teacher-led discussion.

Independent Work – Students could work through the *Think, Do, Discuss* questions of Part 1 and then summarize their findings in their notebooks. For the *Think, Do, Discuss* questions of Part 2, students should use *The Geometer's Sketchpad* or pencil-and-paper construction.

Part 1: Developing the Equation of a Circle

Think, Do, Discuss, p. 577

1. To find the equation of this circle, begin by recalling the distance formula. The distance from $C(2, 3)$ to $P(x, y)$ is 5 units. Substitute this information in the distance formula.

$$\sqrt{(x-2)^2 + (y-3)^2} = 5$$

2. The distance formula has a radical sign. Take steps to “remove” this radical sign. You now have an equation for the circle that is centred at $(2, 3)$ with a radius of 5 units.

$$(x-2)^2 + (y-3)^2 = 25$$

3. What is the equation of the circle whose centre is $(3, -2)$ and whose radius is 4 units?

$$(x-3)^2 + (y+2)^2 = 16$$

4. You can also create a general form for the equation of any circle. Describe the locus of points that are r units from the point $C(h, k)$. Let a point on the circle be $A(x, y)$. Write the formula for the distance from C to A .

The locus of points that are r units from point $C(h, k)$ is a circle with centre C and radius r . The formula for the distance from point $A(x, y)$ on the circle to the centre is

$$\sqrt{(x-h)^2 + (y-k)^2} = CA$$

5. Since you know that the distance CA is the radius, substitute r for CA in the formula in step 4. You now have the general equation for a circle with centre (h, k) and radius r .

The equation for a circle with centre (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.

6. What are the centre and radius of the circle with equation $(x - 1)^2 + (y + 2)^2 = 25$?

The centre is $(1, -2)$ and the radius is 5 units.

7. What is the equation of the circle with centre $(0, 3)$ and radius 8 units?

The equation is $x^2 + (y - 3)^2 = 64$.

8. The equations thus far are in **standard form**. Expand the equation in step 6 to create an equation in the **general form**

$$x^2 + y^2 + 2gx + 2fy + C = 0.$$

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

9. What are the values of g and f for this equation? What do these values tell you?

The values for g and f are -1 and 2 . These values are close to the coordinates of the centre. Changing the signs of both values yields the centre, which is $(-g, -f)$.

10. How would you change an equation in general form to standard form? Change the equation $x^2 + y^2 - 2x - 15 = 0$ to standard form and state the centre and the radius. Use what you learned in grade 10 about completing the square.

The equation in standard form is $(x - 1)^2 + y^2 = 16$. The centre is $(1, 0)$ and the radius is 4.

Part 2: Using Three Points to Find the Equation of a Circle

Think, Do, Discuss, p. 578

1. You know that three points on a circle are $A(5, 4)$, $B(2, 5)$, and $C(-2, 3)$. How could you find the centre and radius of the circle?

Several suggestions are possible: use *The Geometer's Sketchpad*; use a geometric construction with paper, pencil, and compasses; use analytic geometry; or graph the points and estimate the centre and radius.

2. You also know that the centre of the circle must be equidistant from all three points. Take two of the points, such as A and B , and describe the locus of points that are equidistant from points A and B , using geometry or by writing an equation.

Points that are equidistant between A and B lie on the perpendicular bisector of the line segment AB .

Using an equation, first find the midpoint of line segment AB , which is $(3.5, 4.5)$. The perpendicular bisector passes through this point and has slope 3 (the negative reciprocal of the slope of AB). Therefore, the locus or the equation of the perpendicular bisector is $y = 3x - 6$.

3. Describe the locus of points that are equidistant from points B and C , using geometry or by writing an equation.

Points that are equidistant between B and C lie on the perpendicular bisector of line segment BC .

Using an equation, first find the midpoint of line segment BC . The midpoint is $(0, 4)$. The perpendicular bisector passes through this point and has slope -2 (the negative reciprocal of the slope of BC). Therefore, the equation of the perpendicular bisector is $y = -2x + 4$.

4. You can continue to solve this problem geometrically or algebraically. For an algebraic solution, combine the equations that you found in steps 2 and 3. Then find the centre using geometry or a graph. (The graph shown here was created with *The Geometer's Sketchpad*.)

Construct the perpendicular bisector of line segments AB and BC . The intersection point of the perpendicular bisectors is the centre of the circle. This construction is shown in the diagram in the text. The centre is $(2, 0)$.

Using an algebraic solution, find the intersection point of $y = 3x - 6$ and $y = -2x + 4$, which is $(2, 0)$.

5. Find the radius of the circle.

Using *The Geometer's Sketchpad*, simply measure the distance from the centre to one of points A , B , or C . To confirm the radius, measure from the centre to another point.

For an algebraic solution, use the distance formula to find the distance from the centre $(2, 0)$ to one of points A , B , or C .

Additional Example: Functions and Relations

Find the general form of the equation of a circle that has centre $(4, 1)$ and passes through point $(-3, 4)$.

Solution

To find the equation of a circle, you need the centre and the radius. The centre is given. Then find the radius and state the standard form of the equation. Expand the standard form to find the general form of the equation.

The radius is the distance from the centre, $(4, 1)$, to point $(-3, 4)$ on the circle

$$\begin{aligned} r &= \sqrt{[4 - (-3)]^2 + (1 - 4)^2} \\ &= \sqrt{58} \end{aligned}$$

Therefore, the equation of the circle in standard form is

$$(x - 4)^2 + (y - 1)^2 = 58$$

Expanding yields the general form

$$x^2 + y^2 - 8x - 2y - 41 = 0.$$

Practise, Apply, Solve 7.2U

Questions with a Functions and Relations Focus:

1–7, 9, 10, 12, 13, 15, 17, 19, 20

6. Knowledge and Understanding, p. 582

Management Suggestion

This question provides an opportunity for peer tutoring. Students can work in pairs. Partner 1 teaches partner 2 how to do (a) orally, rather than writing and then showing the solution on paper. Then partner 2 teaches partner 1 how to do (b). Let students repeat these steps for the next two questions and then they can work individually on (c) and compare answers.

ASSESSMENT SUGGESTION

Knowledge and Understanding — Informal Diagnostic Assessment

As students are peer tutoring, observe and listen to ensure that students understand how to complete the square and that they recognize the centre and the radius.

Achievement Chart Connection

Students change equations in general form to equations in standard form.

Level 3 Performance

Students should express the equations correctly and identify the centre and the radius of each circle.

9. Communication, p. 582

Management Suggestion

Each student could write a description and then share it with the person sitting next to her or him to see if the description makes sense. Through peer editing, students will make the necessary corrections.

Achievement Chart Connection

Students must describe the family of circles that have the same x - and y -coordinates.

Level 3 Performance

Solutions should include the correct equation for the family of circles whose centres have the same x - and y -coordinates and a diagram, sketch, or graph of this family of circles. As well, students should explain the following:

- All circles are the same size.
- The centres of the circles lie on the line $y = x$.
- There are an infinite number of such circles.

10. Application, p. 582

Management Suggestion

Students could hand in the solutions to this question the following day after it is assigned. Assess solutions to see if students understand and can apply the basic concepts of finding the equation of a circle.

Achievement Chart Connection

Students are expected to find the equation of a circle given different information. This question is unlike the examples. In each case, students must select and use an appropriate strategy.

Level 3 Performance

Solutions should include accurate answers and clear solutions. For many of the solutions, students should draw a diagram. The solutions should also integrate narrative and mathematics so that the solution is readable.

15. Check Your Understanding, p. 583**Management Suggestion**

Partners could exchange their answers to (a) for feedback and peer editing. They could also compare answers to (b). Take up the solutions to (c) in a whole-class discussion.

17. Thinking, Inquiry, Problem Solving, p. 583**Management Suggestion**

Students could use a think-pair-share strategy to solve this question. After they have attempted this question on their own, they can work in pairs to share and discuss their solutions. Ask a few pairs to share their solutions with the class.

Achievement Chart Connection

Students have to select various tools to solve this problem. They need to combine their graphical and analytical skills and use their knowledge of circles and lines.

Level 3 Performance

Responses might include

- an appropriate plan that is carried out and supported
- a graph or diagram of the situation
- a justification of the solution

The Chapter Problem—Designing a Bridge

By the end of this section, students should explore equations of circles that will satisfy the criteria of the Chapter Problem. Students should test many circle equations.

The Chapter Problem Answers are on p. 7-48 of the *Teacher Resource*.

Challenge 1

Remind students that they should find the equation of the circle that was created by shining the flashlight on the wall.

Challenge 2

Remind students that they should test whether the curve they have chosen is a circle.

The Challenge Answers are on p. 7-51 of the *Teacher Resource*.