

7.4U The Ellipse

Specific Expectations

- LC1.03U determine an equation to represent a described locus [e.g., determine the equation of the locus of points equidistant from $(-2, 7)$ and $(5, 4)$]
- LC1.04U construct geometric models to represent the locus definitions of the conics
- LC1.05U determine equations for conics from their locus definitions, by hand for simple particular cases [e.g., determine the equation of the locus of points the sum of whose distances from $(-3, 0)$ and $(3, 0)$ is 10]
- LC2.01U identify the standard forms for the equations of parabolas, circles, ellipses, and hyperbolas having centres at $(0, 0)$ and at (h, k)
- LC3.02U pose and solve problems drawn from a variety of applications involving conics, and communicate the solutions with clarity and justification (Sample problem: A parabolic antenna is 320 m wide at a distance of 50 m above its vertex. Determine the distance above the vertex of the focus of the antenna.)

Evidence of Learning

Students should be able to

- determine the equation of an ellipse centred at the origin given its locus definition
- find the equation of an ellipse centred at $(0, 0)$ given information about the major or minor axes, foci, vertices, x - or y -intercepts, and so on
- determine the equation of an ellipse centred at (h, k) given information about the values of a , b , or c
- draw the graph of an ellipse given in standard form centred either at $(0, 0)$ or at (h, k)
- solve problems involving the application of ellipses

Text Page 585

Time Required Functions and Relations:

75 min

Materials and Equipment

- string, tape, or tacks
- large graph paper

Overview

This section introduces students to the standard equation of the ellipse. They begin by graphing an ellipse using familiar graphing techniques. Then they analyze the graph to learn about the different features of the ellipse. In Part 2, they explore the locus definition for an ellipse through geometric and algebraic modelling.

Management Options

Groups – Group students in pairs to work on the *Think, Do, Discuss* questions of Part 1. Then, as a class, discuss and compare responses. Individuals can explore drawing an ellipse in Part 2 using the suggested manipulatives and then work in pairs on the *Think, Do, Discuss* questions.

Independent Work – Assign the *Think, Do, Discuss* questions of Part 1 for homework the day before the lesson. Spend a few minutes taking up the answers to these questions and then let groups explore Part 2 during class.

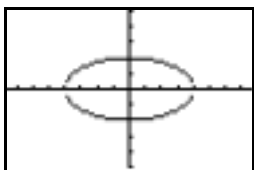
Teacher Demonstration – Display the graph of the equation in *Think, Do, Discuss* question 1 of Part 1 on an overhead and add features of the ellipse as the class works through the other questions. For Part 2, let students explore drawing the ellipse with string. Together, work through the *Think, Do, Discuss* questions in Part 2.

Part 1: Stretching a Circle

Think, Do, Discuss, p. 585

1. Graph $x^2 + 4y^2 = 16$ and describe the graph.

The x -intercepts are ± 4 and the y -intercepts are ± 2 .



2. What is the shape of the graph? What are the x -intercepts? the y -intercepts? Along which axis is the ellipse wider, the x -axis or the y -axis?

The graph is elliptical or oval in shape. The x -intercepts are ± 4 and the y -intercepts are ± 2 . The ellipse is wider along the x -axis.

3. When an ellipse is centred about the origin, the line segment between the x -intercepts and the line segment between the y -intercepts are called the **axes** of the ellipse. The longer segment is the **major axis**, and the shorter axis is the **minor axis**. Label the major axis and the minor axis on your graph.

For the above ellipse, the major axis is along the x -axis and the minor axis is along the y -axis.

4. What is the length of the major axis?

The major axis is 8 units long.

5. Divide both sides of the equation in step 1 by 16. Discuss these aspects of the new equation.

$$\text{The new equation is } \frac{x^2}{16} + \frac{y^2}{4} = 1.$$

The x -intercepts and y -intercepts are the square roots of the denominators.

The length of the major axis is twice the value of the positive x -intercept and the length of the minor axis is twice the value of the positive y -intercept.

6. For the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, predict the x - and y -intercepts. Either draw a graph or use algebra to confirm your prediction. Where is the major axis? What is the length of the major axis?

The x - and y -intercepts are ± 3 and ± 2 , respectively. The major axis is along the x -axis and its length is 6 units.

7. The **standard form** of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. What are the properties of the graph of this equation?

The x -intercepts are $\pm a$ and the y -intercepts are $\pm b$. The length of the major axis is $2|a|$ and the length of the minor axis is $2|b|$.

8. What would be different about the graph of $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a > b$?

In this case, the major axis lies along the y -axis and the minor axis lies along the x -axis.

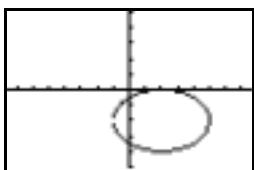
9. All of the above ellipses are centred about the origin. Recalling what you know about circles and given that an ellipse is a “stretched circle,” find the centre of the

$$\text{ellipse } \frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1.$$

The centre is $(1, -2)$.

10. It would be difficult to find the x- and y-intercepts of the ellipse in step 9, but it is not difficult to find the lengths of the major and minor axes. Find the lengths of the major and the minor axes. Sketch a graph of the ellipse in step 9.

The length of the major axis is 6 since $a = 3$, and the length of the minor axis is 4 since $b = 2$.



EXTENSION

Checking with Technology

Ask students to create the equation of an ellipse and to name its x- and y-intercepts. They should then rearrange the equation and use graphing technology to graph the equation. Are their predictions correct? If using graphing calculators, did they remember to “square” the axes?

Part 2: The Locus Definition of an Ellipse

Think, Do, Discuss, p. 586

1. Explain how the diagram on the right illustrates the locus definition of the ellipse.

The length of the string is fixed. Therefore, the sum of the distance from one fixed point to the pencil and the distance from the other fixed point to the pencil is constant. This sum is, in fact, the length of the string.

2. Describe how you would construct an ellipse so that the major axis would be horizontal. How would you construct an ellipse with a vertical major axis?

To construct an ellipse with a horizontal major axis, place the two fixed points on a

horizontal line. To construct an ellipse with a vertical major axis, place the fixed points on a vertical line.

3. Imagine that the ellipse in the diagram is constructed with a piece of string that is 10 units long. What is the relationship between PF_1 and PF_2 ?

$$PF_1 + PF_2 = 10$$

4. What is the length of the major axis for any ellipse?

The length of the major axis is $2|a|$.

5. Show that the length of the string is equal to the length of the major axis by moving $P(x, y)$ to point A_2 .

If P is on A_2 , you can see that

length of the string

$$= F_1F_2 + 2F_2A_2$$

$$= F_1F_2 + A_1F_1 + F_2A_2$$

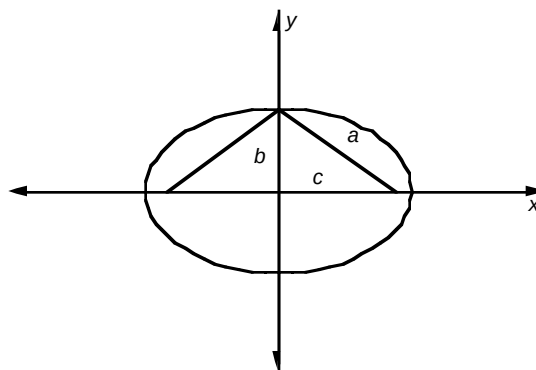
$$= A_1A_2$$

A_1A_2 is the distance between the vertices.

6. Move $P(x, y)$ to point B_2 . What is the length of F_1B_2 ?

$$F_1B_2 = a$$

7. Let a be the distance from the centre of the ellipse to one of the **vertices** of the major axis, A_2 . (Compare this distance to one of the focal radii.) Let b be the distance from the centre to one of the **intercepts** of minor axis, B_2 , and let c be the distance from the fixed point F_2 to the centre. Use your diagram to show that $a^2 = b^2 + c^2$.



Using this diagram, we can see that $\triangle B_2OF_2$ is a right triangle. Therefore, by the Pythagorean theorem,

$$a^2 = b^2 + c^2$$

8. For the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, determine a and b and use these values to determine c . What are the coordinates of the foci for this ellipse? What is the length of string that you would need to draw this ellipse?

For the given ellipse, $a = 5$ and $b = 4$. Since $a^2 = b^2 + c^2$, $c = 3$. Therefore, the foci are $(-3, 0)$ and $(3, 0)$. The length of string needed to draw this ellipse is 10 units.

Additional Example: Functions and Relations

Using the locus definition of an ellipse, find the equation of an ellipse with foci $(0, -4)$ and $(0, 4)$. The sum of the focal radii is 12.

Solution

The sum of the focal radii is constant for any point $P(x, y)$ on the ellipse. In this case, the sum of the focal radii is 12. Therefore, by the distance formula,

$$\sqrt{x^2 + (y + 4)^2} + \sqrt{x^2 + (y - 4)^2} = 12$$

Isolate one radical and then square both sides.

$$\begin{aligned} x^2 + (y - 4)^2 &= 144 - 24\sqrt{x^2 + (y - 4)^2} + x^2 + y^2 - 8y + 16 \end{aligned}$$

Expand the left side.

$$\begin{aligned} x^2 + y^2 + 8y + 16 &= 144 - 24\sqrt{x^2 + (y - 4)^2} + x^2 + y^2 - 8y + 16 \end{aligned}$$

Simplify.

$$16y - 144 = -24\sqrt{x^2 + (y - 4)^2}$$

Divide by 8.

$$2y - 18 = -3\sqrt{x^2 + (y - 4)^2}$$

Square both sides.

$$4y^2 - 72y + 324 = 9(x^2 + y^2 - 8y + 16)$$

Expand.

$$4y^2 - 72y + 324 = 9x^2 + 9y^2 - 72y + 144$$

Simplify.

$$9x^2 + 5y^2 = 180$$

Divide by 180 to get the equation of the ellipse in standard form,

$$\frac{x^2}{20} + \frac{y^2}{36} = 1$$

Practise, Apply, Solve 7.4U

Questions with a Functions and Relations Focus:

1–8, 11, 13, 15, 16, 18–20

3. Knowledge and Understanding, p. 591

Management Suggestion

Assign this question for homework and then ask each group of four to present their solution on large graph paper.

Achievement Chart Connection

Students change simple equations of ellipses to standard form and identify and interpret the values of a , b , and c . Students must sketch graphs of ellipses given their equations.

Level 3 Performance

Solutions should include the accurate calculations of a and b and the application of $a^2 + b^2 = c^2$ to find c and the coordinates of the foci. Sketches of the graphs should show a , b , and c .

7. Communication, p. 592

Management Suggestion

Students could answer this question orally and then graph the ellipse to verify their verbal descriptions.

Achievement Chart Connection

Students must read and then interpret information about an ellipse to describe the ellipse and justify their descriptions.

Level 3 Performance

Descriptions should mention that the major axis is 98 units long and that it is along the x -axis. Since $c = 49$ and $a = 50$, the value of b will be much smaller, about 9.95. Thus, the ellipse will be wide and “short.”

13. Thinking, Inquiry, Problem Solving, p. 593

Management Suggestion

Ask volunteers to present their solutions to the class on the overhead or board so that all students can assess their own work and edit it, if needed. Alternatively, students could submit their work for comment or assessment.

Achievement Chart Connection

Students are expected to explore the problem, create a plan, and then determine an algebraic model for the situation.

ASSESSMENT SUGGESTION

Problem Solving

Have students submit their solutions for question 13 for assessment. Use the component *Creation of a model* of the generic rubric for Thinking, Inquiry, and Problem Solving for assessing. You could also assess communication skills using the component *Integration of narrative and mathematical forms* of the generic rubric for Communication. For both generic rubrics, see Assessment and Evaluation in the *Teacher Resource*.

Level 3 Performance

Responses might include

- the standard equation for an ellipse centred at the origin
- substitution of points in the equation
- solutions for a and b
- the final equation of ellipse

18. Check Your Understanding, p. 594

Management Suggestion

Students could manipulate a piece of string to explore how the shape of an ellipse changes as they move the foci. Perhaps they will see that a circle occurs when the two foci are in the same place, that is, the centre of the circle. Then the value of a is the same as the radius of the circle. Ask them what the corresponding values for b and c are.

19. Application, p. 594

Management Suggestion

Students could begin to work on this problem in groups of three or four. Then students could work individually, going back to the group to compare solutions.

Achievement Chart Connection

Students are expected to apply concepts to familiar and unfamiliar situations.

Level 3 Performance

Solutions should contain

- an accurate diagram
- an equation of the arch based on the given information
- a graph of the arch and the dimensions of the trucks

ASSESSMENT SUGGESTION

Teamwork

As students work in groups, assess their ability to work in teams using the related generic rubric in the Generic Rubrics for Learning Skills. See Assessment and Evaluation in the *Teacher Resource*.

The Chapter

Problem—Designing a Bridge

By the end of this section, students should explore equations of ellipses that will satisfy the criteria in the Chapter Problem. Students should test many ellipse equations.

The Chapter Problem Answers are on p. 7-48 of the *Teacher Resource*.

Challenge 1

Remind students that they should be able to

- see if the flashlight's reflector's shape is an ellipse. If so, what is the equation and what is the focal point?
- find the equation of the ellipse that was created by shining the flashlight on the wall

Challenge 2

Remind students that they should be able to

- test whether the curve they have chosen is an ellipse
- trace the curve on graph paper and choose a few points to substitute in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The Challenge Answers are on p. 7-51 of the *Teacher Resource*.