

7.9U The Hyperbola

Specific Expectations

- LC1.04U construct geometric models to represent the locus definitions of the conics
- LC1.05U determine equations for conics from their locus definitions, by hand for simple particular cases [e.g., determine the equation of the locus of points the sum of whose distances from $(-3, 0)$ and $(3, 0)$ is 10]
- LC2.01U identify the standard forms for the equations of parabolas, circles, ellipses, and hyperbolas having centres at $(0, 0)$ and at (h, k)
- LC2.03U determine the key features (e.g., the centre or the vertex, the focus or foci, the asymptotes, the lengths of the axes) of a conic whose equation is given in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$, by hand in simple cases (e.g., $x^2 + 9y^2 - 6x + 36y - 36 = 0$)
- LC2.04U sketch the graph of a conic whose equation is given in the form $ax^2 + by^2 + 2gx + 2fy + c = 0$
- LC2.05U illustrate the conics as intersections of planes with cones, using concrete materials or technology
- LC3.01U describe the importance, within applications, of the focus of a parabola, an ellipse, or a hyperbola (e.g., all incoming rays parallel to the axis of a parabolic antenna are reflected through the focus; the planets move in elliptical orbits with the sun at one of the foci)
- LC3.02U pose and solve problems drawn from a variety of applications involving conics, and communicate the solutions with clarity and justification (Sample problem: A parabolic antenna is 320 m wide at a distance of 50 m above its vertex. Determine the distance above the vertex of the focus of the antenna)

Evidence of Learning

Students should be able to

- determine the equation of a hyperbola from a locus definition
- determine the equation of a hyperbola given its key features
- determine the key features of a hyperbola given its equation
- solve problems that apply the equation and/or features of a hyperbola
- sketch the graph of a hyperbola by hand and using graphing software

Text Page 609

Time Required Functions and Relations: 75 min

Materials and Equipment

- *The Geometer's Sketchpad* (optional)
- graphing calculator or graphing software

Overview

In this section, students are introduced to the hyperbola. The initial development of the equation of a hyperbola is based on the locus definition. Then students explore the equation of the hyperbola and the key features of hyperbolas. They will also extend their knowledge of hyperbolas through applications.

MANAGEMENT SUGGESTION

You could lead students through developing the equation of the hyperbola from a locus definition (the beginning of Part 1). Then students could work in pairs on the *Think, Do, Discuss* questions. Finally, you could develop the asymptotes by demonstrating Example 1 in Part 2.

Management Options

Groups – Partners could do Part 1 by taking turns explaining each line of the equation's development in the text. Students' understanding will increase if they verbalize this development. Each partner could work on the *Think, Do, Discuss* questions, conferring as needed. Demonstrate Part 2 or direct students to read Example 1 in pairs.

Teacher Demonstration – Lead students through the *Think, Do, Discuss* questions of Part 1, developing the equation of a hyperbola using a locus definition. Then students can do the *Think, Do, Discuss* questions using the think-pair-share strategy. Demonstrate Part 2.

Independent Work – Students could read the development of the equation of the hyperbola and write the steps in their notebooks, which helps to reinforce the concepts and procedures. Then they could answer the *Think, Do, Discuss* questions and record their answers in their notebooks. Next students could work through the example of finding asymptotes in Part 2.

Part 1: Using the Locus Definition to Find the Equation of a Hyperbola

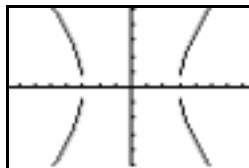
Think, Do, Discuss, p. 611

1. This equation, $\frac{x^2}{9} - \frac{y^2}{16} = 1$, looks similar to the equation of an ellipse. Compare this equation to the equation of an ellipse. What is different and what is the same?

Each aspect of the equations is similar, except for the operation sign, which is subtraction in the hyperbola's equation and addition in the ellipse's equation.

2. Determine both the x- and y-intercepts of $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Draw the graph by finding six other points. You also could use graphing technology, such as Zap-a-Graph, or rearrange the equation so that you can enter it into a graphing calculator.

The x-intercepts are ± 3 and there are no y-intercepts.



3. The foci of this hyperbola are $(-5, 0)$ and $(5, 0)$. Then $c = 5$. If $\frac{x^2}{9} - \frac{y^2}{16} = 1^2$ is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, state the values of a and b.

$a = 3$ and $b = 4$

4. Determine a relationship between a, b, and c.
 $c^2 = a^2 + b^2$
5. The segment between the **vertices** is called the **transverse axis**. Find the length of the transverse axis.

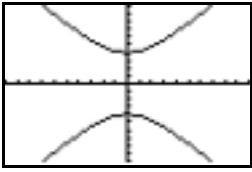
The transverse axis is 6 units long.

6. Consider the other line of symmetry. The **conjugate axis** is perpendicular to the transverse axis and passes through the **centre** of the hyperbola. This axis is the segment between $(-b, 0)$ and $(b, 0)$. What is its length?

The conjugate axis is 8 units long.

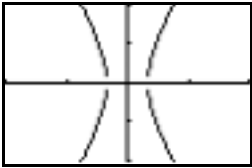
7. Predict what would change if you were to graph $\frac{x^2}{9} - \frac{y^2}{16} = -1$. What would be the x- or y-intercepts? Draw the graph to confirm your prediction. In this case, the equation is in the form $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$. State the values of a, b, and c and state the foci of this hyperbola.

There would be no x -intercepts, and the y -intercepts are ± 4 . $a = 4$, $b = 3$, and $c = 5$. The foci are $(0, 5)$ and $(0, -5)$.

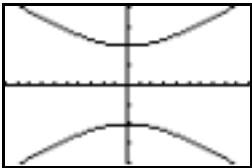


8. Create a table or use graphing technology to graph $\frac{x^2}{4} - \frac{y^2}{36} = 1$ and $\frac{x^2}{25} - \frac{y^2}{4} = -1$. Describe the graphs, stating the x - and y -intercepts.

The first graph has x -intercepts ± 2 and no y -intercepts.



The second graph has y -intercepts ± 2 and no x -intercepts.



9. Using your knowledge of transformations, describe the graph of $\frac{(x-1)^2}{4} - \frac{y^2}{36} = 1$, without graphing the equation. Verify your answer with technology.

The graph of this relation is a hyperbola. The horizontal transverse axis is 4 units long and the conjugate axis is 12 units long. The hyperbola's centre is $(1, 0)$.

10. Describe the graph of $\frac{(x+2)^2}{25} - \frac{(y-1)^2}{4} = 1$. Verify your answer with technology.

The graph is a hyperbola. The horizontal transverse axis is 10 units long and the conjugate axis is 4 units long. The hyperbola's centre is $(-2, 1)$.

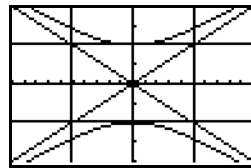
Part 2: Giving Shape to a Hyperbola

Consolidate Your Understanding, p. 612

1. Given $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, determine the general equations of the asymptotes by drawing a rectangle based on the values of a and b .

The general equations are $y = -\left(\frac{b}{a}\right)x$ and

$$y = \left(\frac{b}{a}\right)x.$$



Additional Example: Functions and Relations

Find the equation of a hyperbola with asymptotes $y = 2x$ and $y = -2x$, vertices on the y -axis, and foci $(0, 2\sqrt{5})$ and $(0, -2\sqrt{5})$.

Solution

Since the vertices are on the y -axis and the foci are centred about the origin, the equation is $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$. Since the asymptotes are $y = 2x$ and $y = -2x$, the value of a is twice the value of b . Let $a = 2b$. Then

$$\begin{aligned}c^2 &= a^2 + b^2 \\(2\sqrt{5})^2 &= a^2 + (2a)^2 \\20 &= 5a^2 \\a &= 2\end{aligned}$$

Therefore, $a = 2$ and $b = 4$ and the equation is $\frac{x^2}{16} - \frac{y^2}{4} = -1$.

Practise, Apply, Solve 7.9U**Questions with a Functions and Relations Focus:**

1–10, 12, 14, 16–20

4. Knowledge and Understanding, p. 615**Management Suggestion**

This question has various different cases to consider, for example, where the centre is at the origin and where the centre is not at the origin. In some cases, students will have to divide the equation to rewrite the equation in standard form. Therefore, assign this question to be completed, and also take up the solutions, before the end of class.

Achievement Chart Connection

Students must demonstrate they recognize the centre and the vertices from the equation of a hyperbola in standard form.

Level 3 Performance

Students can easily identify the centre and vertices from an equation in standard form. They should also recognize what first must be done to an equation that is not in standard form.

9. Communication, p. 616**Management Suggestion**

Students could compare their descriptions of the hyperbola and properties with a partner. By doing so, they can refine their own descriptions.

Achievement Chart Connection

Students are expected to use diagrams in the solution to this problem. They also will have to clearly describe the properties of this equilateral hyperbola.

Level 3 Performance

Students should adequately describe all of the properties, including the following:

- Each asymptote makes a 45° angle with the axes.
- The rectangle that helps shape the hyperbola is a square.
- The equation in general form will have the same coefficients of x^2 and y^2 .

Solutions should also include a diagram.

ASSESSMENT SUGGESTION**Application**

Application not only deals with the selection of an appropriate tool but also deals with the proper “fit” of that tool. Look for this fit in student solutions for question 17. Do they know which information gives them values of a , b , or c ? And do they know how to use that information?

17. Application, p. 617**Management Suggestion**

Students could either do this question as a homework exercise or in groups in class. Suggest that a pair or group present their solution to the class.

Achievement Chart Connection

Students are expected to apply their knowledge of the locus definition of hyperbolas to find the equation of this particular hyperbola. If you review Example 4 with students, then the context of this particular application will be somewhat familiar.

Level 3 Performance

Students should include a diagram and recognize that the 30 km represents the difference of the distances between the foci and that the distance between the foci is 100 km. Using this information, they can build an equation for the hyperbola.

18. Check Your Understanding, p. 617**Management Suggestion**

Students will probably use a diagram similar to the one in Part 2 of this section. Extend their thinking by asking them to demonstrate with an example that a point on the given asymptotes will not lie on the hyperbola. Ask a student to present his or her example and solution on the board.

19. Thinking, Inquiry, Problem Solving, p. 617**Management Suggestion**

Have students work in small groups on large graph paper to determine a solution to the specific case. Then assign the more general case for homework to allow students time to work through the problem. Have some students present the solution to the general case the next day in class.

Achievement Chart Connection

Students are expected to explore this algebraic question through an example and then generalize the solution.

Level 3 Performance

Students should be able to determine the asymptotes of the specific case by drawing a diagram. Students should make some progress in finding the asymptotes for the general case. They may need to try another specific example before moving to the general case.