
Check Your Understanding: Answers

1. *Describe three different forms for the equation of a quadratic function. Explain the importance and uses of each form. Describe the algebraic methods for moving from one form to another.*

The equation of a quadratic function may be written in

standard form, $f(x) = ax^2 + bx + c$

factored form, $f(x) = a(x - p)(x - q)$

vertex form, $f(x) = a(x - h)^2 + k$

The standard form shows the y -intercept for the graph of the function. For a profit function, the y -intercept is the initial fixed cost.

The factored form shows the x -intercepts. For a profit function, the x -intercepts are the break-even points.

The vertex form shows the vertex. The y -coordinate of the vertex is the maximum, or minimum, value of the function, and the x -coordinate of the vertex is the value of x that produces the maximum or minimum value. For a profit function, the x -coordinate of the vertex is the number of items that generate the maximum revenue and the y -coordinate is the maximum revenue.

Move from

- standard form to factored form by factoring
 - standard form to vertex form by completing the square or by evaluating $\left[-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right]$
 - factored form to standard form by expanding
 - factored form to vertex form by finding $h = \frac{(p+q)}{2}$ and $f(h)$
 - vertex form to standard form by expanding
 - vertex form to factored form by expanding and then factoring
2. *What shape is the graph of a quadratic function? What information do the coordinates of the vertex of the graph provide? How can you determine the vertex coordinates when the equation is in standard form? in factored form?*

The graph of a quadratic function is a parabola.

The y -coordinate of the vertex is the maximum, or minimum, value of the function, and the x -coordinate of the vertex is the x -value that produces the maximum, or minimum, value.

For an equation in standard form, the x -coordinate of the vertex is $-\frac{b}{2a}$ and the y -coordinate is

$$f\left(-\frac{b}{2a}\right).$$

For an equation in factored form, the x -coordinate of the vertex is $\frac{(p+q)}{2}$. The y -coordinate is

$$f\left[\frac{(p+q)}{2}\right].$$

3. How are the zeros of a function related to the graph of the function? How many zeros does a quadratic function have? Describe how you would determine the number of zeros from the equation of a quadratic function in **(a)** factored form, **(b)** vertex form, and **(c)** standard form.

The zeros occur where the graph crosses the x -axis.

A quadratic function may have no, one, or two zeros.

- (a) For $f(x) = a(x-p)(x-q)$, there are two zeros, $x = p$ and $x = q$. If $f(x) = a(x-p)^2$, there is one zero at $x = p$.
- (b) For $f(x) = a(x-h)^2 + k$, there are
- two zeros if a and k have opposite signs
 - no zeros if a and k have the same signs
 - one zero if $k = 0$.
- (c) For $f(x) = ax^2 + bx + c$, there are
- two zeros if $b^2 - 4ac > 0$
 - one zero if $b^2 - 4ac = 0$
 - no zeros if $b^2 - 4ac < 0$
4. Describe the sets of numbers that make up the set of real numbers. What are complex numbers and how are they related to real numbers? What are complex conjugates and how are they connected to quadratic equations?

The set of real numbers contains natural numbers, $\mathbf{N}$, $\{1, 2, 3, 4, \dots\}$,

whole numbers, $\mathbf{W}$, $\{0, 1, 2, 3, \dots\}$,

integers, $\mathbf{I}$, $\{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$,

rational numbers, $\mathbf{Q}$, $\left\{\frac{a}{b} : a, b \in \mathbf{I}, b \neq 0\right\}$, and

irrational numbers, $\mathbf{Q}$, which cannot be expressed as $\left\{\frac{a}{b} : a, b \in \mathbf{I}, b \neq 0\right\}$.

A complex number is written in the form $a + bi$, where a and b are real numbers and i is a non-real number such that $i^2 = -1$. The set of complex numbers contains all real numbers since any real number can be written as $a + 0i$.

Complex conjugates are complex numbers in the form $a + bi$ and $a - bi$. A quadratic equation with no real roots has two complex conjugate roots.

5. How do you add or subtract complex numbers? How do you multiply complex numbers? What happens when you add complex conjugates? What happens when you multiply complex conjugates? How do you divide complex numbers? Give an example for each.

Add or subtract complex numbers by combining like terms, that is, the real terms and then the imaginary terms. For example,

$$\begin{aligned}(3 - 4i) + (-1 + 2i) &= [3 + (-1)] + (-4i + 2i) \\ &= 2 - 2i\end{aligned}$$

Use the distributive law when multiplying complex numbers. For example,

$$\begin{aligned}(4 - i)(-2 + 3i) &= -8 + 12i + 2i - 3i^2 \\ &= -8 - 3(-1) + 14i \\ &= -5 + 14i\end{aligned}$$

Sums and products of complex conjugates are always entirely real numbers. For example,

$$(3 + 4i) + (3 - 4i) = 6$$

$$\begin{aligned}(3 + 4i)(3 - 4i) &= 9 - 16i^2 \\ &= 25\end{aligned}$$

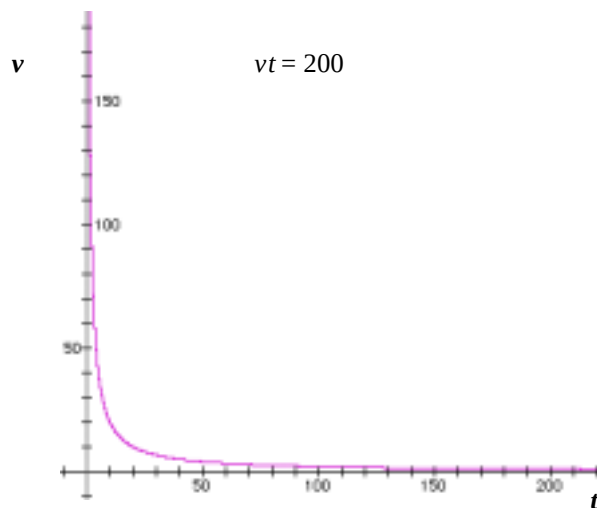
Divide complex numbers by multiplying the numerator and denominator by the conjugate of the denominator. For example,

$$\begin{aligned}\frac{3+i}{4-3i} &= \frac{3+i}{4-3i} \times \frac{4+3i}{4+3i} \\ &= \frac{12+9i+4i+3i^2}{16-9i^2} \\ &= \frac{9+13i}{25} \\ &= \frac{9}{25} + \frac{13}{25}i\end{aligned}$$

6. *What is inverse variation? Give an example of two quantities that vary inversely and write an equation that describes the relationship between them. State the domain and the range and sketch the graph of this relation.*

When one quantity increases as the other decreases and the product of the two quantities remains constant, then the quantities vary inversely. For example, for a journey of a fixed distance of 200 km, the speed and the time taken for the journey are inversely related. The equation

$v \times t = 200$, or $v = \frac{200}{t}$, describes the relationship between speed, v , and time, t . In this context, the domain and the range are positive real numbers.



7. How can you use the graph of a function to obtain the graph of its reciprocal function? Explain the steps using an example.

Sketch the graph of $f(x) = \frac{1}{x^2 - 9}$ from the graph of $f(x) = x^2 - 9$ as follows:

- $f(x) = x^2 - 9$ has zeros at $x = \pm 3$, so $\frac{1}{f(x)}$ has vertical asymptotes $x = 3$ and $x = -3$.
 - $f(x) = x^2 - 9$ has value 1 when $x = \pm\sqrt{10}$ and $f(x) = x^2 - 9$ has value -1 when $x = \pm\sqrt{8}$, so the graph of $\frac{1}{f(x)}$ passes through $(\sqrt{10}, 1)$, $(-\sqrt{10}, 1)$, $(\sqrt{8}, -1)$, and $(-\sqrt{8}, -1)$.
 - When $f(x)$ is close to 1, $\frac{1}{f(x)}$ is also close to 1.
 - When $f(x)$ is close to -1 , $\frac{1}{f(x)}$ is also close to -1 .
 - $f(x) = x^2 - 9$ is positive when $x < -3$ and when $x > 3$, so $\frac{1}{f(x)}$ is positive in the same intervals.
 - $f(x) = x^2 - 9$ is negative when $-3 < x < 3$, so $\frac{1}{f(x)}$ is negative in the same interval.
 - The y -intercept for $\frac{1}{f(x)}$ is $(0, -\frac{1}{9})$ since the y -intercept for $f(x) = x^2 - 9$ is $(0, -9)$.
 - As $f(x) = x^2 - 9$ gets larger, $\frac{1}{f(x)}$ gets smaller. As $f(x) = x^2 - 9$ gets smaller, $\frac{1}{f(x)}$ gets larger.
8. What is a rational function? How are the graphs of reciprocal and rational functions different from the graphs of the other functions you have studied? Describe their features with examples.

A rational function is a quotient of two polynomial functions. For example, $f(x) = \frac{x-3}{x+1}$ is a rational function.

- The domain of a rational function or a reciprocal function consists of all real numbers *except* those that make the denominator equal 0. For $f(x) = \frac{x-3}{x+1}$, the domain is all real numbers except -1 .
- The zeros of a rational function are the zeros of the function in the numerator, provided the rational function is in reduced form. For $f(x) = \frac{x-3}{x+1}$, the zero is $(3, 0)$.
- Reciprocal functions have horizontal and vertical asymptotes.
- Rational functions may have vertical, horizontal, or oblique asymptotes.
- For a rational function, vertical asymptotes occur at the zeros of the function in the denominator, provided the rational function is in reduced form. Both the reciprocal function $f(x) = \frac{1}{x+1}$ and the rational function $f(x) = \frac{x-3}{x+1}$ have the vertical asymptote $x = -1$.
- A horizontal asymptote occurs at a break in the range, which happens when the highest power of x in the numerator is less than or equal to the highest power of x in the denominator. The rational function $f(x) = \frac{x-3}{x+1}$ does have a horizontal asymptote, $y = 1$, because the powers of

x in both the numerator and denominator are equal. The reciprocal function $f(x) = \frac{1}{x+1}$ also has a horizontal asymptote, $y = 0$ (the x -axis), because we can imagine that the numerator is x^0 .

- An oblique asymptote occurs when the graph of a function approaches a straight, imaginary line with a defined, nonzero slope as x becomes very large or very small. An oblique asymptote will exist when the highest power of x in the numerator is exactly one more than the highest power of x in the denominator. The rational function $f(x) = \frac{x^2-1}{x}$ has oblique asymptote $y = x$.

9. Explain how you would simplify a rational expression. Include a discussion of restrictions. Why must you state the restrictions before reducing an expression?

Simplify a rational expression by dividing the numerator and denominator by any nonzero factor of the numerator and denominator. Then remove the common factors. Exclude any values of the variables that would make the denominator 0, and since the denominator may change at each step, it is important to state these restrictions on the variables at each step, including the step before reducing an expression.

10. When you add or subtract rational expressions, what steps do you take to find the lowest common denominator? Explain with an example.

Factor the denominator of each term. To find the lowest common denominator (LCD), multiply together each different factor the minimum number of times it occurs. For example, to find the

LCD of $\frac{x+1}{x^2-8x+16}$, $\frac{2x}{x+4}$ and $\frac{x^2-1}{x^2-16}$, factor the denominators.

The factors are $(x-4)(x-4)$, $(x+4)$, and $(x+4)(x-4)$.

The LCD is $(x-4)^2(x+4)$.

11. Write an example of a polynomial expression, state its degree, and explain how you determined the degree. What special names are given to polynomials with one, two, or three terms? Is your polynomial one of these special types? Explain.

$2 - 3x + x^2 - 5x^3 + 2x^6$ is an example of a polynomial expression. The degree of this polynomial is 6, since the highest power of any term is 6.

A polynomial with one term is a monomial, a polynomial with two terms is a binomial, and a polynomial with three terms is a trinomial. The above example is not one of these special types since it has more than three terms.