
Challenge 1: Solution Guide

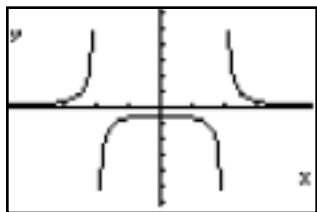
This is a good pre-calculus challenge suitable for students in either course.

To answer this challenge, students will need to experiment, using graphing technology and their knowledge of functions from Chapters 3 and 4. In addition, since they have not yet learned about oblique asymptotes, they will need to do some research for part (c). Chapter problem question CP16 introduces an Average Profit function that has an oblique asymptote. Also, you could direct students to the functions in Question 10 of the Practise, Apply, Solve exercise for section 4.11, which also have oblique asymptotes. The functions in question 10 are stated in the form in which the oblique asymptote equation is apparent. Students would need to make this connection between the equation and the graph.

The following are a sample of the many possible solutions.

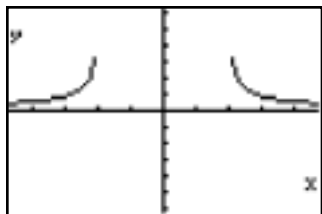
(a) (i) The reciprocals of many quadratic functions like the ones graphed in section 4.6 have graphs with two parallel vertical asymptotes. These graphs occupy all three regions of the xy -plane.

For example, the graph for $f(x) = \frac{1}{x^2 - 4}$ is shown (for values of x from -4.7 to 4.7 , and values of y from -3.1 to 3.1).



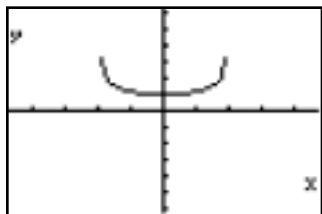
Vertical asymptotes have equations $x = \pm 2$

(ii) To get a graph that occupies only two regions you need a function that is not defined between two values. The square root function $f(x) = \sqrt{x^2 - 4}$ is not defined when $-2 < x < 2$. The graph for $f(x) = \frac{1}{\sqrt{x^2 - 4}}$ is shown (for the same values of x and y as in (i)).



Vertical asymptotes have equations $x = \pm 2$

(iii) For a graph that only occupies one region, again we need a function with a restricted domain like $f(x) = \sqrt{4-x^2}$, which is only defined when $-2 \leq x \leq 2$. The graph of $f(x) = \frac{1}{\sqrt{4-x^2}}$ will only lie between vertical asymptotes $x = 2$ and $x = -2$. Vertical asymptotes have equations $x = \pm 2$



(b) One approach to graphs with two parallel horizontal asymptotes is to exchange x and y in the equation for a function with two parallel vertical asymptotes; that is consider the inverse function. The inverse for $f(x) = \frac{1}{x^2-4}$ is not a function unless the domain of $f(x)$ is restricted to positive

values of x only, or to negative values of $x < -\frac{1}{4}$. Then each inverse function only has one

horizontal asymptote. However, it is possible to define a piecewise function $f(x) = \begin{cases} \sqrt{\frac{1}{x}+4}, & x < -\frac{1}{4} \\ -\sqrt{\frac{1}{x}+4}, & x > 0 \end{cases}$

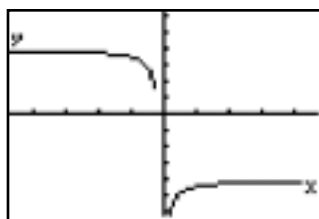
which has two parallel horizontal asymptotes and occupies two regions.

(ii) Enter the function in Y1 and Y2 as shown, using 2nd TEST menu to obtain the inequality signs.

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Plot1 Plot2 Plot3
Y1=√((1/X)+4)
(X<(-1/4))
Y2=-√((1/X)+4)
(X>0)
Y3=
Y4=
Y5=

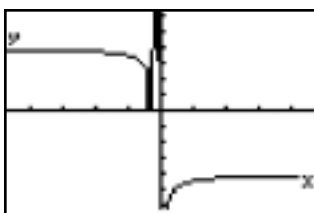
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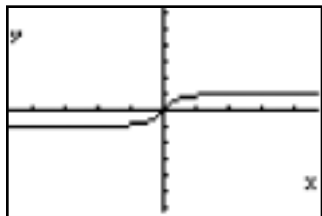
Horizontal asymptotes have equations $y = \pm 2$. There is no part of the graph above the upper asymptote.

(i) Once you allow piecewise functions the challenge is less challenging! To obtain a function with a graph in all three regions simply add another “piece.”

$$f(x) = \begin{cases} \sqrt{\frac{1}{x}+4}, & x < -\frac{1}{3} \\ 12x+5, & -\frac{1}{3} < x < 0 \\ -\sqrt{\frac{1}{x}+4}, & x < 0 \end{cases}$$

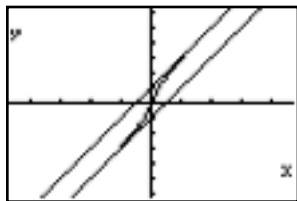
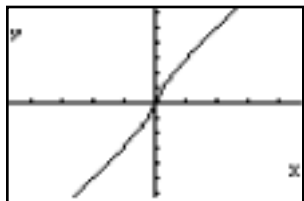


(iii) The function $f(x) = \frac{x}{\sqrt{4x^2 + 1}}$ has a graph that lies entirely in one region between parallel horizontal asymptotes $y = \pm \frac{1}{2}$.



(c) Adding an x , the function $f(x) = \frac{x}{\sqrt{4x^2 + 1}}$, lies entirely between oblique asymptotes $y = \pm \frac{1}{2}$.

(iii)



A similar approach can be used with the graphs in (b) (i) and (ii) to obtain graphs in two and three regions.

If students are introduced to absolute value functions, these provide alternate solutions for (b) and (c). Examples:

(b) 2 horizontal asymptotes (i) 3 regions $f(x) = \frac{|x-5|}{x-5} \left(1 + \frac{1}{x}\right)$ Asymptotes $y = \pm 1$

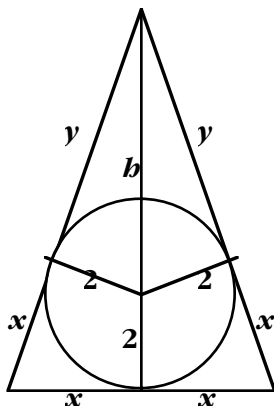
(ii) 2 regions $f(x) = \frac{|x-2|}{x-2} + \frac{1}{\sqrt{x^2-4}}$, asymptotes $y = \pm 1$

(c) 2 oblique asymptotes (i) 3 regions $f(x) = \frac{x^2 + 5|x| - 5}{x}$, asymptotes $y = x \pm 5$

(ii) 2 regions $f(x) = \frac{x^2 + 5|x| + 5}{x}$, asymptotes $y = x \pm 5$.

Challenge 2: Solution Guide

The challenge here for students will be to set up the problem. They need to draw on their measurement and geometry knowledge acquired in grade 9. However, expect students to develop an algebraic model for the area, rather than simply using a spreadsheet, as they would have done in grade 9.



To label this diagram, students need to know that tangents from an external point to a circle are equal in length and that a tangent is perpendicular to the radius at the point of tangency.

Then students can use the Pythagorean theorem to find the side lengths of the triangle.

Solve for y .

$$h^2 = 2^2 + y^2$$

$$h^2 = 4 + y^2$$

$$y = \sqrt{h^2 - 4}$$

Solve for x .

$$x^2 + (h + 2)^2 = (x + y)^2$$

$$x^2 + h^2 + 4h + 4 = x^2 + 2xy + y^2$$

$$h^2 + 4h + 4 = 2xy + y^2$$

Find the area of the largest triangle.

$$\begin{aligned} A &= \frac{1}{2} Bh \\ &= \frac{1}{2} (x + x)(h + 2) \\ &= \frac{1}{2} (2x)(h + 2) \\ &= x(h + 2) \end{aligned}$$

But $y = \sqrt{h^2 - 4}$, so

$$h^2 + 4h + 4 = 2xy + y^2$$

$$h^2 + 4h + 4 = 2x\sqrt{h^2 - 4} + (\sqrt{h^2 - 4})^2$$

$$h^2 + 4h + 4 = 2x\sqrt{h^2 - 4} + h^2 - 4$$

$$4h + 8 = 2x\sqrt{h^2 - 4}$$

$$2h + 4 = x\sqrt{h^2 - 4}$$

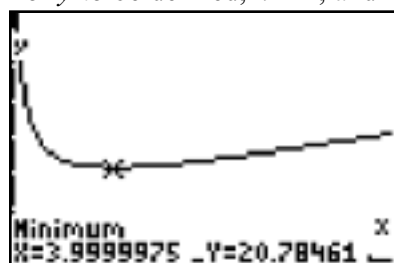
$$x = \frac{2h + 4}{\sqrt{h^2 - 4}}$$

Substitute the expression for x .

$$\begin{aligned}
 A &= x(h+2) \\
 &= \frac{2h+4}{\sqrt{h^2-4}}(h+2) \\
 &= \frac{2(h+2)}{\sqrt{h^2-4}}(h+2) \\
 &= \frac{2(h+2)^2}{\sqrt{h^2-4}}
 \end{aligned}$$

At this point, students should consider the domain of this area function.

For y to be defined, $h \geq 2$, and for x and A to be defined, $h > 2$.



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WINDOW
Xmin=2
Xmax=9.4
Xscl=1
Ymin=0
Ymax=55
Yscl=1
Xres=1

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The area is at a minimum when $h = 4$.

Then $x = \frac{6}{\sqrt{3}}$ and $y = 2\sqrt{3}$.

Therefore, the triangle is equilateral and each side is $\frac{12}{\sqrt{3}}$, or about 6.93, units long.

CHAPTER 4: RUBRIC FOR CHALLENGE 2

Assessment Category	Scores and Criteria			
	Passable level of achievement – below provincial standard LEVEL 1	Moderate level of achievement – below, but approaching provincial standard LEVEL 2	High level of achievement – at provincial standard LEVEL 3	Very high to outstanding level of achievement – above provincial standard LEVEL 4
Knowledge and Understanding				
Demonstration of understanding of a concept	Answer attempted but work shows limited understanding of the incircle, tangency, area as a function, and the domain of this function.	Work shows some understanding of the incircle, tangency, area as a function, and the domain of this function.	Solution shows good understanding of the incircle, tangency, area as a function, and the domain of this function.	Detailed solution shows thorough understanding of the concepts.
Performance of an algorithm by hand, mentally, or using technology	Student obtains an area function using an incorrect procedure, or makes many errors in Pythagorean computations. Student uses technology to find dimensions for minimum area with limited success.	Student attempts to use Pythagorean theorem to relate dimensions and generate an area function, but work may contain inaccuracies. Student uses technology to find dimensions for minimum area with some success.	Student uses Pythagorean theorem to relate dimensions and area function generated, with most computations generally correct. Dimensions for minimum area successfully found using technology.	Area function is generated efficiently, accurately, and work is verified or supported. Dimensions for minimum area successfully found using a variety of technologies.

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