

form,  $f(x) = ax^2 + bx + c$ , provides the  $y$ -intercept  $(0, c)$ . However, if information about the maximum or minimum value of a quadratic function is required, the vertex form of its equation will provide this. In vertex form,  $f(x) = a(x - p) + q$ , the coordinates of the vertex  $(p, q)$  locate the position of the maximum or minimum point on the parabola. The equation of a quadratic function in standard form may be transformed to vertex form by the method of “completing the square.” The steps are as follows.

Given  $f(x) = ax^2 + bx + c$ :

- Factor ‘a’ out of the first two terms.

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

- Add and subtract the square of half  $\frac{b}{a}$  inside the bracket.

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c$$

- Group the perfect square terms and move the subtracted value outside the brackets, remembering to multiply by ‘a’.

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] - a\left(\frac{b}{2a}\right)^2 + c$$

- Factor the perfect square.

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c$$

### Practise, Apply, Solve 4.2, page 314

- (a) i.

$$f(x) = x^2 - 5x - 8$$

$$= \left(x^2 - 5x + \frac{25}{4} - \frac{25}{4}\right) - 8$$

$$= \left(x^2 - 5x + \frac{25}{4}\right) - \frac{25}{4} - \frac{32}{4}$$

$$= \left(x - \frac{5}{2}\right)^2 - \frac{57}{4}$$

Vertex has coordinates  $\left(\frac{5}{2}, -\frac{57}{4}\right)$ .

- Function has a minimum value since  $a > 0$ .

- $x = \frac{5}{2}$  produces this minimum value.

- The minimum value is  $-\frac{57}{4}$  (or  $-14.25$ ).

- (b) i.

$$f(x) = -2x^2 + 16x - 17$$

$$= -2(x^2 - 8x) - 17$$

$$= -2(x^2 - 8x + 16 - 16) - 17$$

$$= -2(x^2 - 8x + 16) - (-2)(-16) - 17$$

$$= -2(x - 4)^2 + 15$$

Vertex has coordinates  $(4, 15)$ .

- Function has a maximum value since  $a < 0$ .

- Maximum occurs when  $x = 4$ .

- Maximum value is 15.

- (c) i.

$$f(x) = -4x^2 - 6x - 5$$

$$= -4\left(x^2 + \frac{3}{2}x\right) - 5$$

$$= -4\left(x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right) - 5$$

$$= -4\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) - (-4)\left(\frac{9}{16}\right) - 5$$

$$= -4\left(x + \frac{3}{4}\right)^2 - \frac{11}{4}$$

Vertex is at  $\left(-\frac{3}{4}, -\frac{11}{4}\right)$  (or  $(-0.75, -2.75)$ ).

- ii The function has a maximum value since  $a < 0$ .

- iii. Maximum value occurs where  $x = -\frac{3}{4}$  (or  $-0.75$ ).

- iv. Maximum value of the function is  $-\frac{11}{4}$  (or  $-2.75$ ).

- (d) i.

$$f(x) = -\frac{1}{2}x^2 + 2x - 9$$

$$= -\frac{1}{2}(x^2 - 4x) - 9$$

$$= -\frac{1}{2}(x^2 - 4x + 4 - 4) - 9$$

$$= -\frac{1}{2}(x^2 - 4x + 4) - \left(-\frac{1}{2}\right)(4) - 9$$

$$= -\frac{1}{2}(x - 2)^2 - 7$$

Vertex is at  $(2, -7)$ .

- ii. Function has a maximum value since  $a < 0$ .

- iii. Maximum value occurs where  $x = 2$ .

- iv. Maximum value of the function is  $-7$ .

2. (a)  $f(x) = (x - 3)(x - 8)$  has  $x$ -intercepts at  $(3, 0)$  and  $(8, 0)$ .

The vertex is halfway between these intercepts and so its

$x$ -coordinate is  $\frac{3+8}{2}$  or  $5.5$ . The corresponding  $y$ -coordinate

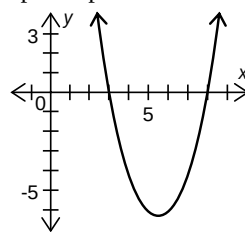
is found by substituting  $5.5$  for  $x$  in the function equation.

$$f(5.5) = (5.5 - 3)(5.5 - 8)$$

$$= (2.5)(-2.5)$$

$$= -6.25$$

Vertex has coordinates  $(5.5, -6.25)$ . Since  $a > 0$ , the parabola opens up.



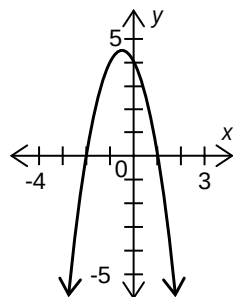
- (b)  $f(x) = -2(x - 1)(x + 2)$  has  $x$ -intercepts at  $(1, 0)$  and

$(-2, 0)$ . The  $x$ -coordinate of the vertex is  $\frac{1+(-2)}{2}$  or  $-0.5$ .

The  $y$ -coordinate is  $f(-0.5)$ .

$$\begin{aligned} f(-0.5) &= -2(-0.5 - 1)(-0.5 + 2) \\ &= -2(-1.5)(1.5) \\ &= 4.5 \end{aligned}$$

Vertex has coordinates  $(-0.5, 4.5)$ .  
Since  $a < 0$ , the parabola opens down.

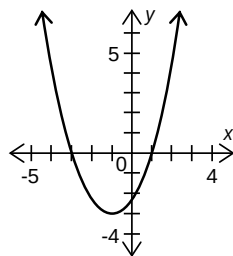


$$\begin{aligned} \text{(c) } f(x) &= -0.7(1.1 - x)(3.2 + x) \\ &= 0.7(x - 1.1)(x + 3.2) \end{aligned}$$

Graph has  $x$ -intercepts at  $(1.1, 0)$  and  $(-3.2, 0)$ .  $x$ -coordinate of vertex is  $\frac{1.1 + (-3.2)}{2}$  or  $-1.05$ . The  $y$ -coordinate of the vertex is  $f(-1.05)$ .

$$\begin{aligned} f(-1.05) &= 0.7(-1.05 - 1.1)(-1.05 + 3.2) \\ &= 0.7(-2.15)(2.15) \\ &= -3.23575 \end{aligned}$$

Vertex has coordinates  $(-1.05, -3.23575)$ .  
Since  $a > 0$ , the parabola opens up.



$$\text{(d) } f(x) = -\frac{5}{6}\left(x - \frac{2}{3}\right)\left(x + \frac{1}{6}\right)$$

Graph has  $x$ -intercepts  $\left(\frac{2}{3}, 0\right)$  and  $\left(-\frac{1}{6}, 0\right)$ .

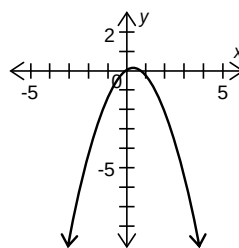
$x$ -coordinate of vertex is  $\frac{\frac{2}{3} + \left(-\frac{1}{6}\right)}{2}$  or  $\frac{1}{4}$ . The  $y$ -coordinate

of the vertex is  $f\left(\frac{1}{4}\right)$ .

$$\begin{aligned} f\left(\frac{1}{4}\right) &= -\frac{5}{6}\left(\frac{1}{4} - \frac{2}{3}\right)\left(\frac{1}{4} + \frac{1}{6}\right) \\ &= -\frac{5}{6}\left(-\frac{5}{12}\right)\left(\frac{5}{12}\right) \\ &= \frac{125}{864} \quad (\approx 0.14) \end{aligned}$$

Vertex has coordinates  $\left(\frac{1}{4}, \frac{125}{864}\right)$ .

Since  $a < 0$ , the parabola opens down.



$$\begin{aligned} \text{3. (a) } f(x) &= 2x^2 - 6x \\ &= 2x(x - 3) \end{aligned}$$

Graph has  $x$ -intercepts at  $(0, 0)$  and  $(3, 0)$ .  $x$ -coordinate of vertex is  $\frac{0 + 3}{2} = 1.5$ .  $y$ -coordinate of vertex is:

$$\begin{aligned} f(1.5) &= 2(1.5)^2 - 6(1.5) \\ &= -4.5 \end{aligned}$$

Vertex is at  $(1.5, -4.5)$ .

Since  $a > 0$ ,  $f(x)$  has a minimum value.

$$\begin{aligned} \text{(b) } h(t) &= -5t^2 + 15t \\ &= -5t(t - 3) \end{aligned}$$

Graph has  $x$ -intercepts at  $(0, 0)$  and  $(3, 0)$ .  $x$ -coordinate of vertex is  $\frac{0 + 3}{2}$  or  $1.5$ .  $y$ -coordinate of vertex is:

$$\begin{aligned} h(1.5) &= -5(1.5)^2 + 15(1.5) \\ &= 11.25 \end{aligned}$$

Vertex is at  $(1.5, 11.25)$ .

Since  $a < 0$ ,  $h(t)$  has a maximum value.

$$\begin{aligned} \text{(c) } s(t) &= -4.9t^2 + 19.6t + 22 \\ x\text{-coordinate of vertex is } -\frac{b}{2a} &= \frac{-19.6}{2(-4.9)} = 2 \end{aligned}$$

$$\begin{aligned} y\text{-coordinate of vertex is } s(2): \\ s(2) &= -4.9(2)^2 + 19.6(2) + 22 \\ &= 41.6 \end{aligned}$$

Vertex is  $(2, 41.6)$ .

Since  $a < 0$ ,  $s(t)$  has a maximum value.

$$\begin{aligned} \text{(d) } f(x) &= 2.5x^2 - 15x - 24 \\ x\text{-coordinate of vertex is } -\frac{b}{2a} &= \frac{15}{2(2.5)} = 3 \end{aligned}$$

$$\begin{aligned} f(3) &= 2.5(3^2) - 15(3) - 24 \\ &= -46.5 \end{aligned}$$

Vertex is at  $(3, -46.5)$ .

Since  $a > 0$ ,  $f(x)$  has a minimum value.

**4. Revenue = Number of items sold  $\times$  Price of each item**

$$\text{(a) i. } R(x) = x(-x + 7)$$

$R(x) = -x^2 + 7x$  is the revenue function.

ii. The graph of  $R(x)$  opens down since  $a < 0$ .  $R(x)$  has a maximum value.  $x$ -coordinate of the vertex for the graph is

$$-\frac{b}{2a} = \frac{-7}{2(-1)} = 3.5$$

Since  $x$  is in thousands, this means that 3500 items must be sold to maximize revenue.

$$\begin{aligned} \text{The price is } p &= -x + 7 \\ &= -3.5 + 7 \\ &= 3.5 \text{ or } \$3.50 \end{aligned}$$

$\therefore$  maximum revenue is  $3500 \times \$3.50 = \$12\,250$ .

$$\text{(b) i. } R(x) = x(-3x + 11)$$

$R(x) = -3x^2 + 11x$  is the revenue function.

ii. x-coordinate of vertex is:

$$-\frac{b}{2a} = \frac{-11}{2(-3)} \doteq 1.833$$

$\therefore$  1833 items sold will maximize revenue. (Note that in real-life terms,  $x$  must be a whole number.)

The price is  $p = -3x + 11$

$$= -3(1.8\dot{3}) + 11$$

$$= 5.5 \text{ or } \$5.50$$

$\therefore$  maximum revenue is  $1833 \times \$5.50 = \$10\,081.50$ .

(c) i.  $R(x) = x(-0.4x + 14)$

$R(x) = -0.4x^2 + 14x$  is the revenue function.

ii. x-coordinate of vertex is:

$$-\frac{b}{2a} = \frac{-14}{2(-0.4)} = 17.5$$

$\therefore$  17 500 items sold will maximize revenue.

The price is  $p = -0.4(17.5) + 14$

$$= 7 \text{ or } \$7$$

$\therefore$  maximum revenue is  $17\,500 \times \$7 = \$122\,500$ .

5.  $P(x) = R(x) - C(x)$

(a) i.  $P(x) = -x^2 + 24x - (12x + 28)$

$$= -x^2 + 24x - 12x - 28$$

$$= -x^2 + 12x - 28$$

ii. x-coordinate of vertex is

$$-\frac{b}{2a} = \frac{-12}{2(-1)} = 6$$

Value of  $x$  that maximizes profit is 6.

(b) i.  $P(x) = -2x^2 + 32x - (14x + 45)$

$$= -2x^2 + 32x - 14x - 45$$

$$= -2x^2 + 18x - 45$$

ii. x-coordinate of vertex is

$$-\frac{b}{2a} = \frac{-18}{2(-2)} = 4.5$$

Value of  $x$  that maximizes profit is 4.5.

(c) i.  $P(x) = -3x^2 + 26x - (8x + 18)$

$$= -3x^2 + 26x - 8x - 18$$

$$= -3x^2 + 18x - 18$$

(ii) x-coordinate of vertex is

$$-\frac{b}{2a} = \frac{-18}{2(-3)} = 3$$

Value of  $x$  that maximizes profit is 3.

(d) i.  $P(x) = -2x^2 + 25x - (3x + 17)$

$$= -2x^2 + 25x - 3x - 17$$

$$= -2x^2 + 22x - 17$$

ii. x-coordinate of vertex is

$$-\frac{b}{2a} = \frac{-22}{2(-2)} = 5.5$$

Value of  $x$  that maximizes profit is 5.5.

6. (a) Revenue = Number of items sold  $\times$  Price of each item

Using  $R(x)$  to denote the revenue function,

$$R(x) = x(-5x + 21)$$

$$= -5x^2 + 21x$$

(b) Using  $P(x)$  to denote the profit function,

$$P(x) = R(x) - C(x)$$

$$= -5x^2 + 21x - (4x + 14)$$

$$= -5x^2 + 21x - 4x - 14$$

$$= -5x^2 + 17x - 14$$

$$(c) P(x) = -5(x^2 - 3.4x) - 14$$

$$= -5(x^2 - 3.4x + 2.89 - 2.89) - 14$$

$$= -5(x^2 - 3.4x + 2.89) + 14.45 - 14$$

$$= -5(x - 1.7)^2 + 0.45$$

The value of  $x$  that will maximize profits is 1.7, which represents 1700 items.

(d) The break-even quantities are given by the x-intercepts for  $P(x)$ . To find these, we set  $P(x) = 0$ . (or use the quadratic formula to obtain the roots)

$$-5x^2 + 17x - 14 = 0$$

$$(-5x + 7)(x - 2) = 0$$

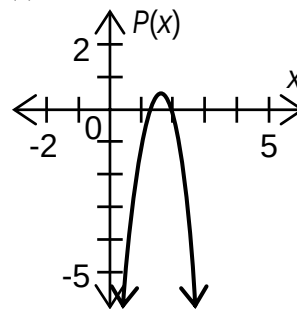
$$-5x + 7 = 0 \text{ or } x - 2 = 0$$

$$7 = 5x \quad x = 2$$

$$1.4 = x$$

The break-even quantities are 1400 items and 2000 items.

(e)



$$7. C(x) = 0.28x^2 - 0.7x + 1$$

The most economical production level is the level at which the cost is a minimum. This will occur at the vertex on the graph of  $C(x)$ . We only need the x-coordinate of the vertex to answer the question. This x-coordinate is:

$$-\frac{b}{2a} = \frac{0.7}{2(0.28)}$$

$$= 1.25$$

$\therefore$  1250 items per hour is the most economical production level.

$$8. h(t) = -4.9t^2 + 6t + 0.6$$

We need both coordinates of the vertex. x-coordinate is:

$$-\frac{b}{2a} = \frac{-6}{2(-4.9)} \doteq 0.612$$

$$\text{y-coordinate is } h(0.612) = -4.9(0.612)^2 + 6(0.612) + 0.6$$

$$\doteq 2.437 \quad (\text{to 3 decimal places})$$

The ball reaches its maximum height when  $t \doteq 0.6$  seconds.

The maximum height reached by the ball is about 2.4 m (to 1 decimal place).

9. Let the numbers be  $x$  and  $y$ . Then  $x + y = 10$

$$\text{so } y = 10 - x$$

The product of the numbers,  $P = xy$

$$= x(10 - x)$$

$$= -x^2 + 10x$$

$$= -(x^2 - 10x)$$

$$= -(x^2 - 10x + 25 - 25)$$

$$= -(x^2 - 10x + 25) + 25$$

$$= -(x - 5)^2 + 25$$

The maximum value that  $P$  can take is 25.

10. Let  $f(x) = 2x^2 - 8x + 9$

Complete the square to find the minimum value possible for  $f(x)$ . We know  $f(x)$  has a minimum value since  $a > 0$ .

$$\begin{aligned} f(x) &= 2(x^2 - 4x) + 9 \\ &= 2(x^2 - 4x + 4 - 4) + 9 \\ &= 2(x^2 - 4x + 4) - 8 + 9 \\ &= 2(x - 2)^2 + 1 \end{aligned}$$

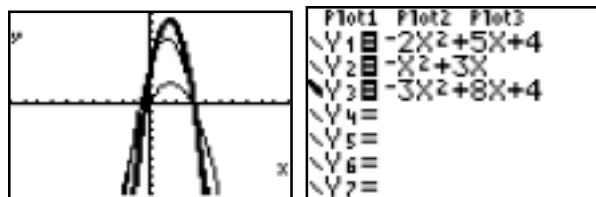
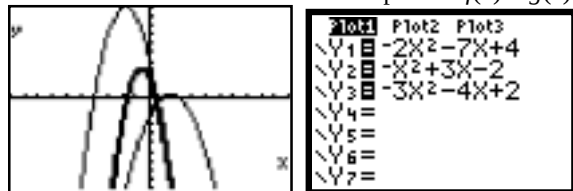
The minimum possible value for  $f(x)$  is 1. Therefore,  $2x^2 - 8x + 9$  can never be less than 1.

11. The function equations for  $f(x)$  and  $g(x)$  are both in the form  $ax^2 + bx + c$  where  $a < 0$ .

Let  $f(x) = a_1x^2 + b_1x + c_1$  and  $g(x) = a_2x^2 + b_2x + c_2$

Then  $f(x) + g(x) = a_1x^2 + b_1x + c_1 + a_2x^2 + b_2x + c_2$   
 $= (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)$

Since this new function is in the form  $ax^2 + bx + c$  and since we know  $a_1 + a_2 < 0$  (since  $a_1 < 0$  and  $a_2 < 0$ ),  $f(x) + g(x)$  is a quadratic function and its graph opens downward. The position of the vertex in relation to those for  $f(x)$  and  $g(x)$  is more difficult to show algebraically. However, if students investigate several examples, using a graphing calculator, they will soon notice that the  $x$ -coordinate of the vertex is always in between the  $x$ -coordinates of the vertices for  $f(x)$  and  $g(x)$ . The  $y$ -coordinate of the vertex for  $f(x) + g(x)$  may be above, in between, or below those for  $f(x)$  and  $g(x)$  depending on the positions of  $f(x)$  and  $g(x)$ . Some examples are shown here. Darkened curves represent  $f(x) + g(x)$ .



12. (a) Let  $x$  represent the number of additional trees planted. Then the total number of apple trees =  $25 + x$   
Yield per tree =  $(500 - 10x)$

Total yield  $Y(x) = (25 + x)(500 - 10x)$

To find the value of  $x$  which provides maximum yield, find the  $x$ -coordinate of the vertex.

Method A

$x$ -intercepts for  $Y(x)$  are  $(-25, 0)$  and  $(50, 0)$ .

$x$ -coordinate of vertex is halfway between  $-25$  and  $50$

$$\frac{-25 + 50}{2} = 12.5$$

Part 2: Functions

Method B

$$\begin{aligned} Y(x) &= -10x^2 + 250x + 12\,500 \\ &= -10(x^2 - 25x) + 12\,500 \\ &= -10(x^2 - 25x + 12.5^2) + 10(12.5)^2 + 12\,500 \\ &= -10(x - 12.5)^2 + 10(12.5)^2 + 12\,500 \end{aligned}$$

$x$ -coordinate of vertex is 12.5.

Since you cannot plant half a tree, 12 additional trees should be planted, ie. a total of 37 trees will maximize the yield.

(b) Revenue = Yield  $\times$  Price per apple

$$\begin{aligned} \therefore R(x) &= Y(x) \times 0.15 \\ &= -1.5x^2 + 37.5x + 1875 \end{aligned}$$

and  $C(x)$  = number of trees  $\times$  cost per tree

$$\begin{aligned} &= (25 + x) \times 6.50 \\ &= 162.5 + 6.5x \end{aligned}$$

Profit = Revenue - Cost

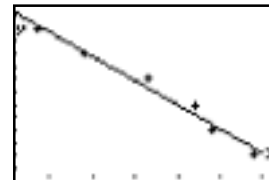
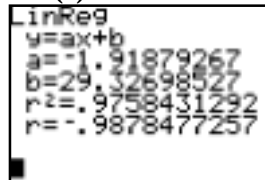
$$\begin{aligned} P(x) &= -1.5x^2 + 37.5x + 1875 - 162.5 - 6.5x \\ &= -1.5x^2 + 31x + 1712.5 \end{aligned}$$

$x$ -coordinate of vertex is:

$$\begin{aligned} -\frac{b}{2a} &= \frac{-31}{2(-1.5)} \\ &= 10.\bar{3} \end{aligned}$$

10 additional trees should be planted to maximize profit, ie. a total of 35 trees will maximize the profit.

13. (a)



Demand function is  $p(x) = -2x + 29$

(b) Revenue  $R(x) = x(-2x + 29)$   
 $= -2x^2 + 29x$

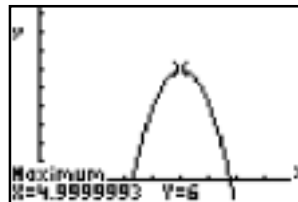
(c) Profit  $P(x) = R(x) - C(x)$   
 $= -2x^2 + 29x - (9x + 44)$   
 $= -2x^2 + 20x - 44$

Completing the square,  $P(x) = -2(x^2 - 10x) - 44$   
 $= -2(x^2 - 10x + 25 - 25) - 44$   
 $= -2(x^2 - 10x + 25) + 50 - 44$   
 $= -2(x - 5)^2 + 6$

Maximum profit occurs when  $x = 5$ .

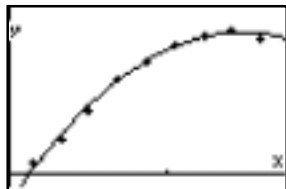
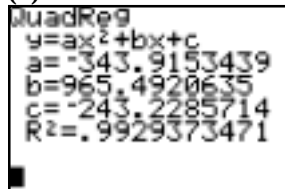
5000 pizzas should be sold to maximize profit.

Or, enter  $P(x)$  in a graphing calculator and use "CALC Maximum" to obtain coordinates of vertex.



14. (a) Looking at the table, the maximum yield is around 44 200 kg/hectare.

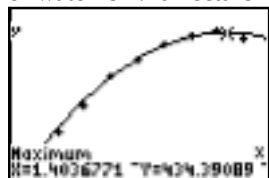
(b)



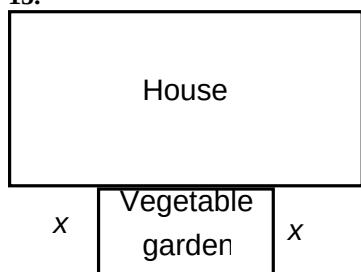
Quadratic regression provides the model

$$y = -344x^2 + 965x - 243.$$

(c) Using the CALC Maximum function, the optimal amount of water is 1.40 hectare-meters.



15.



$$24 - 2x$$

Let the width of the garden be  $x$  m. Then the length is  $24 - 2x$  m.

$$\text{Area of garden} = x(24 - 2x)$$

$$A(x) = -2x^2 + 24x$$

The largest garden is the one with the greatest area. We need to find the value of  $x$  that produces the greatest area.

$$A(x) = -2(x^2 - 12x)$$

$$= -2(x^2 - 12x + 36 - 36)$$

$$= -2(x^2 - 12x + 36) + 72$$

$$= -2(x - 6)^2 + 72$$

Area will be greatest when  $x = 6$ . The dimensions of the garden are  $12 \times 6$  m.

16. Let the number of days which elapse before the steer is sold be  $x$ . After  $x$  days, the steer weighs  $135 + 3.5x$  kg. Market price is  $\$(1.65 - 0.01x)$  per kg. Revenue from sale of steer,  $R(x) = (135 + 3.5x)(1.65 - 0.01x)$ . The cost to keep the steer for  $x$  days is  $\$0.80x$ .

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$\therefore \text{Profit } P(x) = (135 + 3.5x)(1.65 - 0.01x) - 0.80x$$

$$= 222.75 - 1.35x + 5.775x - 0.035x^2 - 0.80x$$

$$= -0.035x^2 + 3.625x + 222.75$$

To find the value of  $x$  that maximizes profit, we need to find the  $x$ -coordinate of the vertex for the graph of  $P(x)$ .

$$x\text{-coordinate of vertex is } -\frac{b}{2a} = \frac{-3.625}{2(-0.035)} \doteq 51.79$$

The steer should be sold after 52 days to maximize profit.

17.  $y = ax^2 + bx = x(ax + b)$

When  $x$  is zero,  $y$  must be zero.  $\therefore$  the graph will always pass through the origin, whatever the values of  $a$  and  $b$ .

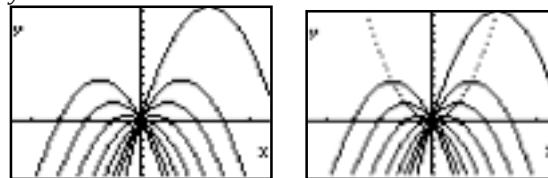
(a) Let  $a = -2$ , and choose values of  $b$  from  $-6$  to  $+8$ .

Convert each equation to vertex form and list vertex coordinates.

| Standard Form    | Vertex Form                                          | (b) Vertex                               |
|------------------|------------------------------------------------------|------------------------------------------|
| $y = -2x^2 - 6x$ | $y = -2(x + 1.5)^2 + 4.5$                            | $(-1.5, 4.5)$                            |
| $y = -2x^2 - 4x$ | $y = -2(x + 1)^2 + 2$                                | $(-1, 2)$                                |
| $y = -2x^2 - 2x$ | $y = -2\left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$ | $\left(-\frac{1}{2}, \frac{1}{2}\right)$ |
| $y = -2x^2 + 0x$ | $y = -2x^2$                                          | $(0, 0)$                                 |
| $y = -2x^2 + 2x$ | $y = -2\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$ | $\left(\frac{1}{2}, \frac{1}{2}\right)$  |
| $y = -2x^2 + 4x$ | $y = -2(x - 1)^2 + 2$                                | $(1, 2)$                                 |
| $y = -2x^2 + 6x$ | $y = -2(x - 1.5)^2 + 4.5$                            | $(1.5, 4.5)$                             |
| $y = -2x^2 + 8x$ | $y = -2(x - 2)^2 + 8$                                | $(2, 8)$                                 |

(c) By inspection, the relationship is  $y = 2x^2$ . (or enter coordinates into lists in a graphing calculator and do a quadratic regression to obtain the equation.)

The vertices all lie on the same parabola with equation  $y = 2x^2$ .



(d),(e) You can repeat the above process with  $a$  left as a variable.

| Standard Form   | Vertex Form                                          | Vertex                                     |
|-----------------|------------------------------------------------------|--------------------------------------------|
| $y = ax^2 - 6x$ | $y = a\left(x - \frac{3}{a}\right)^2 - \frac{9}{a}$  | $\left(\frac{3}{a}, -\frac{9}{a}\right)$   |
| $y = ax^2 - 4x$ | $y = a\left(x - \frac{2}{a}\right)^2 - \frac{4}{a}$  | $\left(\frac{2}{a}, -\frac{4}{a}\right)$   |
| $y = ax^2 - 2x$ | $y = a\left(x - \frac{1}{a}\right)^2 - \frac{1}{a}$  | $\left(\frac{1}{a}, -\frac{1}{a}\right)$   |
| $y = ax^2$      | $y = ax^2$                                           | $(0, 0)$                                   |
| $y = ax^2 + 2x$ | $y = a\left(x + \frac{1}{a}\right)^2 - \frac{1}{a}$  | $\left(-\frac{1}{a}, -\frac{1}{a}\right)$  |
| $y = ax^2 + 4x$ | $y = a\left(x + \frac{2}{a}\right)^2 - \frac{4}{a}$  | $\left(-\frac{2}{a}, -\frac{4}{a}\right)$  |
| $y = ax^2 + 6x$ | $y = a\left(x + \frac{3}{a}\right)^2 - \frac{9}{a}$  | $\left(-\frac{3}{a}, -\frac{9}{a}\right)$  |
| $y = ax^2 + 8x$ | $y = a\left(x + \frac{4}{a}\right)^2 - \frac{16}{a}$ | $\left(-\frac{4}{a}, -\frac{16}{a}\right)$ |

This time the vertex coordinates are related by  $y = -ax^2$ . The vertices are all on the same parabola  $y = -ax^2$ .

18. (a) A quadratic function may be written in standard form,  $f(x) = ax^2 + bx + c$ , in factored form,  $f(x) = a(x - p)(x - q)$  or in vertex form,  $f(x) = a(x - h)^2 + k$

(b) For example, the function  $g(x) = -2(x - 1)(x + 4)$  is in factored form and may be transformed to standard form by expanding the brackets:  $g(x) = -2(x^2 + 3x - 4)$

So  $g(x) = -2x^2 - 6x + 8$  in standard form.

This may then be transformed to vertex form by completing the square.

$$\begin{aligned} g(x) &= -2(x^2 + 3x) + 8 \\ &= -2(x^2 + 3x + 1.5^2 - 1.5^2) + 8 \\ &= -2(x^2 + 3x + 1.5^2) + 2(1.5)^2 + 8 \end{aligned}$$

$$g(x) = -2(x + 1.5)^2 + 12.5 \text{ in vertex form.}$$

(c) In factored form,  $g(x) = -2(x - 1)(x + 4)$  the  $x$ -intercepts and direction of opening are clear to see. For  $g(x)$ , the  $x$ -intercepts are  $(1, 0)$  and  $(-4, 0)$  and the parabola opens downwards, since  $a < 0$ .

In standard form, the  $y$ -intercept and also the direction of opening are clear to see. For  $g(x) = -2x^2 - 6x + 8$ , the  $y$ -intercept is  $(0, 8)$  and the parabola opens downwards as before.

In vertex form, the coordinates of the vertex are clear to see. Also, the maximum or minimum value of the function is given by the  $y$ -coordinate of the vertex. For

$g(x) = -2(x + 1.5)^2 + 12.5$ , the vertex is at  $(-1.5, 12.5)$ . Since  $a < 0$ , the graph opens downwards and  $g(x)$  has a maximum value of 12.5 when  $x = -1.5$ .

(d) Factored form is useful if you need to know the  $x$ -intercepts or zeros of a function. For instance, if the function was a profit function, these would give the break-even quantities.

Standard form is useful if you need the zeros or  $x$ -intercepts but the function equation does not factor. With the equation in standard form, it is easy to use the quadratic formula to obtain the zeros. Again, these might be break-even points. Vertex form is useful if you need to know a maximum or minimum value or where this occurs. For example, you might want to know the maximum height reached by a ball thrown in the air, and also, how long it takes for the ball to reach this maximum height. The maximum height would be given by the  $y$ -coordinate of the vertex, and the time to reach this height by the  $x$ -coordinate.

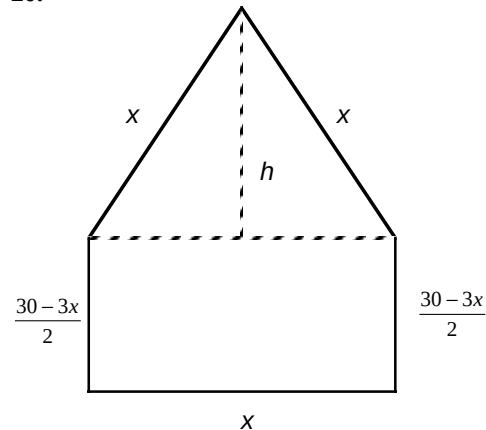
**19.**  $h(t) = -4.9t^2 + v_0t + h_0$

To find the time to reach the maximum height, we need the equation in vertex form.

$$\begin{aligned} h(t) &= -4.9 \left( t^2 - \frac{v_0}{4.9}t \right) + h_0 \\ &= -4.9 \left[ t^2 - \frac{v_0}{4.9}t + \left( -\frac{v_0}{9.8} \right)^2 - \left( -\frac{v_0}{9.8} \right)^2 \right] + h_0 \\ &= -4.9 \left[ t^2 - \frac{v_0}{4.9}t + \left( -\frac{v_0}{9.8} \right)^2 \right] + (4.9) \left( -\frac{v_0}{9.8} \right)^2 + h_0 \\ &= -4.9 \left( t - \frac{v_0}{9.8} \right)^2 + \frac{v_0^2}{19.6} + h_0 \end{aligned}$$

Time to reach maximum height is  $\frac{v_0}{9.8}$  seconds.

**20.**



Let the length of each side of the triangular part of the playground be represented by  $x$  metres. Then two sides of the rectangular part also has length  $x$  metres and the other two sides are each  $\frac{30-3x}{2}$  metres as shown. If  $h$  metres is the

height of the equilateral triangle shown, then, by the Pythagorean Theorem:

$$h^2 + \left( \frac{x}{2} \right)^2 = x^2$$

$$h^2 = x^2 - \frac{x^2}{4}$$

$$h = \frac{\sqrt{3}}{2}x$$

Alternatively, using trigonometry:

$$h = x \sin 60^\circ = \frac{\sqrt{3}}{2}x$$

Area of triangular part of playground is:

$$\frac{1}{2}x \left( \frac{\sqrt{3}}{2}x \right) = \frac{\sqrt{3}}{4}x^2$$

$$\begin{aligned} \text{Total area of playground} &= \text{area of triangle} + \text{area of rectangle} \\ &= \frac{\sqrt{3}}{4}x^2 + x \left( \frac{30-3x}{2} \right) \\ &= \frac{\sqrt{3}}{4}x^2 + 15x - \frac{3x^2}{2} \\ &= \frac{\sqrt{3}-6}{4}x^2 + 15x \end{aligned}$$

Call this  $A(x)$  and put  $A(x)$  in completed square form:

$$\begin{aligned} A(x) &= \frac{\sqrt{3}-6}{4} \left( x^2 + \frac{60}{\sqrt{3}-6} x \right) \\ &= \frac{\sqrt{3}-6}{4} \left[ x^2 + \frac{60}{\sqrt{3}-6} x + \left( \frac{30}{\sqrt{3}-6} \right)^2 - \left( \frac{30}{\sqrt{3}-6} \right)^2 \right] \\ &= \frac{\sqrt{3}-6}{4} \left[ x^2 + \frac{60}{\sqrt{3}-6} x + \left( \frac{30}{\sqrt{3}-6} \right)^2 \right] - \left( \frac{\sqrt{3}-6}{4} \right) \left( \frac{30}{\sqrt{3}-6} \right)^2 \\ &= \frac{\sqrt{3}-6}{4} \left( x + \frac{30}{\sqrt{3}-6} \right)^2 - \frac{225}{\sqrt{3}-6} \end{aligned}$$

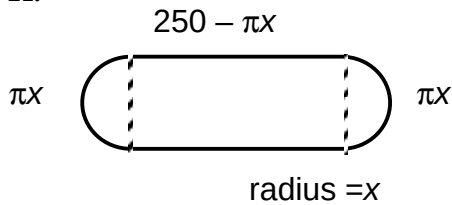
The value of  $x$ , which maximizes  $A(x)$ , is  $\left| -\frac{30}{\sqrt{3}-6} \right| \doteq 7.03$

The pentagonal play area with the maximum area will have 3 sides of length 7.03 m and two of length

$$\frac{30 - 3(7.03)}{2} \doteq 4.46 \text{ m.}$$

The dimensions of the rectangle are 7.03 m  $\times$  4.46 m. The dimensions of the triangle are 7.03 m  $\times$  7.03 m  $\times$  7.03 m.

21.



Let the radius, in metres, of the semicircular ends be represented by  $x$ . Then each semicircular end has length  $\pi x$  m. Since 2 laps is 1000 m, one lap is 500 m and so the straight parts are each  $\frac{500 - 2\pi x}{2} = 250 - \pi x$  m.

(a) Total area contained = Total area of 2 ends + area of rectangular

$$= \pi x^2 + 2x(250 - \pi x)$$

Denoting this area by  $A(x)$ ,

$$A(x) = \pi x^2 + 500x - 2\pi x^2$$

$$= -\pi x^2 + 500x$$

$$= -\pi \left( x^2 - \frac{500}{\pi} x \right)$$

$$= -\pi \left[ x^2 - \frac{500}{\pi} x + \left( -\frac{250}{\pi} \right)^2 - \left( -\frac{250}{\pi} \right)^2 \right]$$

$$= -\pi \left( x - \frac{250}{\pi} \right)^2 + \frac{250^2}{\pi}$$

The value of  $x$ , which makes  $A(x)$  a maximum, is

$\frac{250}{\pi} \doteq 79.58$  m. In this case, the length of the rectangular part of the track is:

$$250 - \pi x = 250 - \pi \left( \frac{250}{\pi} \right) = 0$$

The track is entirely circular with radius 79.58 m. This will satisfy the track coach, since two laps around the track will be

1000 m, but there will be no rectangular field area for the physical education department.

(b) Area of rectangular portion

$$\text{Area of rectangle} = 2x(250 - \pi x)$$

$$= -2\pi x^2 + 500x$$

$$= -2\pi \left( x^2 - \frac{250}{\pi} x \right)$$

$$= -2\pi \left( x^2 - \frac{250}{\pi} x + \frac{125^2}{\pi^2} - \frac{125^2}{\pi^2} \right)$$

$$= -2\pi \left( x^2 - \frac{250}{\pi} x + \frac{125^2}{\pi^2} \right) + 2\pi \left( \frac{125^2}{\pi^2} \right)$$

$$= -2\pi \left( x - \frac{125}{\pi} \right)^2 + \frac{2(125)^2}{\pi}$$

The maximum value of this area occurs when  $x = \frac{125}{\pi}$ .

In this case, the length of the straight part of the track will be

$$250 - \pi \left( \frac{125}{\pi} \right) = 125 \text{ m. This is long enough for the track team}$$

to be able to run the 100 m dash.

### Practise, Apply, Solve 4.3, page 325

1. (a)



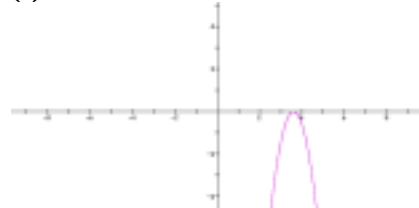
$f(x) = 2(x - 1.2)^2 - 5$  has 2 zeros.

(b)



$f(x) = -1.5(x + 3)^2 + 1.2$  has 2 zeros.

(c)



$f(x) = -4(x - 3.6)^2$  has one zero.

(d)



$f(x) = 0.5(x - 3.2)^2 + 2$  has no zeros.