

Call this $A(x)$ and put $A(x)$ in completed square form:

$$\begin{aligned} A(x) &= \frac{\sqrt{3}-6}{4} \left(x^2 + \frac{60}{\sqrt{3}-6} x \right) \\ &= \frac{\sqrt{3}-6}{4} \left[x^2 + \frac{60}{\sqrt{3}-6} x + \left(\frac{30}{\sqrt{3}-6} \right)^2 - \left(\frac{30}{\sqrt{3}-6} \right)^2 \right] \\ &= \frac{\sqrt{3}-6}{4} \left[x^2 + \frac{60}{\sqrt{3}-6} x + \left(\frac{30}{\sqrt{3}-6} \right)^2 \right] - \left(\frac{\sqrt{3}-6}{4} \right) \left(\frac{30}{\sqrt{3}-6} \right)^2 \\ &= \frac{\sqrt{3}-6}{4} \left(x + \frac{30}{\sqrt{3}-6} \right)^2 - \frac{225}{\sqrt{3}-6} \end{aligned}$$

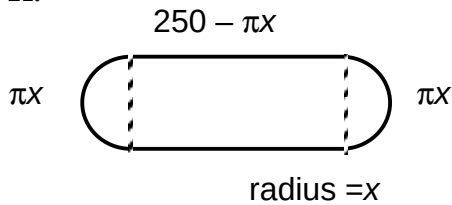
The value of x , which maximizes $A(x)$, is $\left| -\frac{30}{\sqrt{3}-6} \right| \doteq 7.03$

The pentagonal play area with the maximum area will have 3 sides of length 7.03 m and two of length

$$\frac{30 - 3(7.03)}{2} \doteq 4.46 \text{ m.}$$

The dimensions of the rectangle are 7.03 m $\times$ 4.46 m. The dimensions of the triangle are 7.03 m $\times$ 7.03 m $\times$ 7.03 m.

21.



Let the radius, in metres, of the semicircular ends be represented by x . Then each semicircular end has length πx m. Since 2 laps is 1000 m, one lap is 500 m and so the straight parts are each $\frac{500 - 2\pi x}{2} = 250 - \pi x$ m.

(a) Total area contained = Total area of 2 ends + area of rectangular

$$= \pi x^2 + 2x(250 - \pi x)$$

Denoting this area by $A(x)$,

$$A(x) = \pi x^2 + 500x - 2\pi x^2$$

$$= -\pi x^2 + 500x$$

$$= -\pi \left(x^2 - \frac{500}{\pi} x \right)$$

$$= -\pi \left[x^2 - \frac{500}{\pi} x + \left(-\frac{250}{\pi} \right)^2 - \left(-\frac{250}{\pi} \right)^2 \right]$$

$$= -\pi \left(x - \frac{250}{\pi} \right)^2 + \frac{250^2}{\pi}$$

The value of x , which makes $A(x)$ a maximum, is

$\frac{250}{\pi} \doteq 79.58$ m. In this case, the length of the rectangular part of the track is:

$$250 - \pi x = 250 - \pi \left(\frac{250}{\pi} \right) = 0$$

The track is entirely circular with radius 79.58 m. This will satisfy the track coach, since two laps around the track will be

1000 m, but there will be no rectangular field area for the physical education department.

(b) Area of rectangular portion

$$\text{Area of rectangle} = 2x(250 - \pi x)$$

$$= -2\pi x^2 + 500x$$

$$= -2\pi \left(x^2 - \frac{250}{\pi} x \right)$$

$$= -2\pi \left(x^2 - \frac{250}{\pi} x + \frac{125^2}{\pi^2} - \frac{125^2}{\pi^2} \right)$$

$$= -2\pi \left(x^2 - \frac{250}{\pi} x + \frac{125^2}{\pi^2} \right) + 2\pi \left(\frac{125^2}{\pi^2} \right)$$

$$= -2\pi \left(x - \frac{125}{\pi} \right)^2 + \frac{2(125)^2}{\pi}$$

The maximum value of this area occurs when $x = \frac{125}{\pi}$.

In this case, the length of the straight part of the track will be

$$250 - \pi \left(\frac{125}{\pi} \right) = 125 \text{ m. This is long enough for the track team}$$

to be able to run the 100 m dash.

Practise, Apply, Solve 4.3, page 325

1. (a)



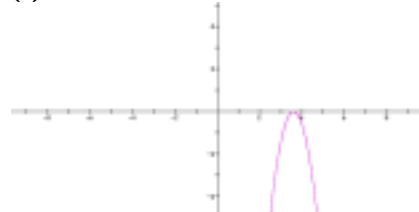
$f(x) = 2(x - 1.2)^2 - 5$ has 2 zeros.

(b)



$f(x) = -1.5(x + 3)^2 + 1.2$ has 2 zeros.

(c)



$f(x) = -4(x - 3.6)^2$ has one zero.

(d)



$f(x) = 0.5(x - 3.2)^2 + 2$ has no zeros.

2. (a) $f(x) = -2(x-3)^2 - 4$, $a = -2$, $k = -4$
 Since a and k have the same sign, $f(x) = -2(x-3)^2 - 4$ has no zeros.
- (b) $f(x) = -3(x+2.5)^2 + 3.2$, $a = -3$, $k = +3.2$
 Since a and k have opposite signs, $f(x) = -3(x+2.5)^2 + 3.2$ has two zeros.
- (c) $f(x) = 4(x+2)^2$, $a = 4$, $k = 0$
 Since $k = 0$, $f(x) = 4(x+2)^2$ has one zero.
- (d) $f(x) = 6.2(x-3.5)^2 + 4.4$, $a = +6.2$, $k = 4.4$
 Since a and k have the same sign, $f(x) = 6.2(x-3.5)^2 + 4.4$ has no zeros.

3. (a) $f(x) = 2x^2 + 3x + 1$

$$b^2 - 4ac = 3^2 - 4(2)(1) \\ = 9 - 8 \\ = 1$$

Since $b^2 - 4ac > 0$, $f(x)$ has two zeros.

(b) $f(x) = -3x^2 + 5x - 3$

$$b^2 - 4ac = 5^2 - 4(-3)(-3) \\ = 25 - 36 \\ = -11$$

Since $b^2 - 4ac < 0$, $f(x)$ has no zeros.

(c) $f(x) = 5x^2 - 4x + 2$

$$b^2 - 4ac = (-4)^2 - 4(5)(2) \\ = 16 - 40 \\ = -24$$

Since $b^2 - 4ac < 0$, $f(x)$ has no zeros.

4. (a) $3x^2 - 2x + 1 = 0$

$$b^2 - 4ac = (-2)^2 - 4(3)(1) \\ = 4 - 12 \\ = -8$$

Since $b^2 - 4ac < 0$, this equation has no real roots.

(b) $-2(x-1.3)^2 + 5 = 0$,

$$a = -2, k = 5$$

Since a and k have opposite signs, the equation has two roots.

(c) $-3(x-4)(x+1) = 0$

$$x = 4 \text{ or } x = -1$$

The equation has two real roots.

(d) $-7(x-4)^2 = 0$, $a = -7$, $k = 0$

Since $k = 0$, this equation has one root.

5. (a) $f(x) = x^2 - 6x + 7$

$$b^2 - 4ac = (-6)^2 - 4(1)(7) \\ = 36 - 28 \\ = 8$$

Since $b^2 - 4ac > 0$, $f(x)$ has two zeros. The graph of $f(x)$ intersects the x -axis in two points.

(b) $f(x) = 9 - x^2 = (3-x)(3+x)$

$f(x)$ has two zeros. The graph of $f(x)$ intersects the x -axis in two points.

(c) $f(x) = (4+x)^2$

The graph of $f(x)$ touches the x -axis. $f(x)$ has one zero.

(d) $f(x) = -2(x-1)^2 - 1$, $a = -2$, $k = -1$

Since a and k have the same sign, $f(x)$ has no zeros. The graph of $f(x)$ does not intersect the x -axis.

6. The profit functions for each machine are:

Machine A:

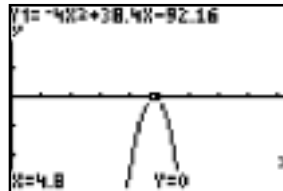
$$P_A(x) = (-4x + 42.5)x - (4.1x + 92.16) \\ = -4x^2 + 38.4x - 92.16$$

Machine B:

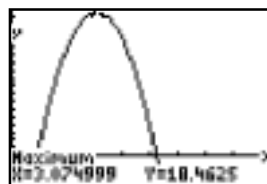
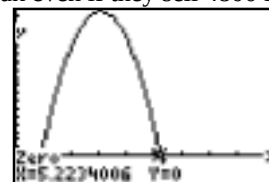
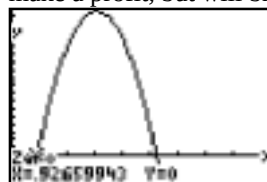
$$P_B(x) = (-4x + 42.5)x - (17.9x + 19.36) \\ = -4x^2 + 24.6x - 19.36$$

Machine C:

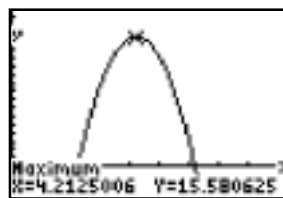
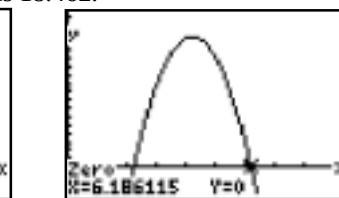
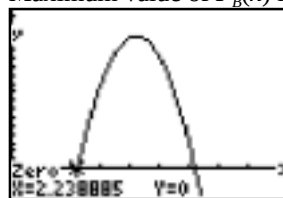
$$P_C(x) = (-4x + 42.5)x - (8.8x + 55.4) \\ = -4x^2 + 33.7x - 55.4$$



Using Machine A, the graph of the profit function has only one x -intercept. $P_A(x) = 0$ when $x = 4.8$. The company will not make a profit, but will break even if they sell 4800 items.



Using Machine B, the graph of $P_B(x)$ has two x -intercepts. $P_B(x) = 0$ when $x = 0.927$ and $x = 5.223$. The company breaks even at 927 items, then profit increases to a maximum at 3075 items before falling off to a break-even point at 5223 items. Maximum value of $P_B(x)$ is 18.462.



Using Machine C, the graph of $P_C(x) = 0$ when $x = 2.239$ and $x = 6.186$. The company breaks even at 2239 items, then profit increases to a maximum at 4213 items before falling off to a break-even point at 6186 items. Maximum value of $P_C(x)$ is 15.581.

According to the models given, by using Machine B, the company will begin to make a profit sooner than with either of Machine A or C. Also the maximum profit possible is higher with Machine B than with A or C. Recommend use of Machine B.

7. $f(x) = kx^2 - 4x + k$
 If $f(x)$ has no zeros, $b^2 - 4ac < 0$.
 $(-4)^2 - 4(k)(k) < 0$
 $16 - 4k^2 < 0$
 $16 < 4k^2$
 $k < -2$ or $k > 2$
 $f(x)$ has no zeros when $k < -2$ or $k > 2$.

8. $g(x) = 4x^2 - 4x + m$
 If $g(x)$ has one zero, $b^2 - 4ac = 0$.
 $(-4)^2 - 4(4)(m) = 0$
 $16 - 16m = 0$
 $16 = 16m$
 $1 = m$
 $g(x)$ has exactly one zero when $m = 1$.

9. $3x^2 + 4x + k = 0$
 $b^2 - 4ac = 4^2 - 4(3)(k)$
 $= 16 - 12k$
 For no roots, $16 - 12k < 0$
 $16 < 12k$
 $\frac{4}{3} < k$
 If $k > \frac{4}{3}$, the equation has no roots.
 For one root, $16 - 12k = 0$

$$k = \frac{4}{3}$$

If $k = \frac{4}{3}$, the equation has exactly one root.
 For two roots, $16 - 12k > 0$
 $16 > 12k$
 $\frac{4}{3} > k$

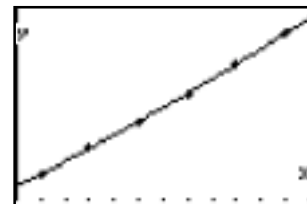
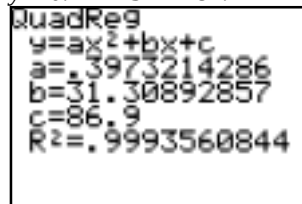
If $k < \frac{4}{3}$, the equation has two roots.

10. $f(x) = x^2 - kx + k + 8$
 If the graph of $f(x)$ touches the x -axis, $f(x)$ has exactly one zero, and $b^2 - 4ac = 0$.
 $b^2 - 4ac = (-k)^2 - 4(1)(k + 8)$
 $= k^2 - 4k - 32$
 So $k^2 - 4k - 32 = 0$
 $(k - 8)(k + 4) = 0$
 $k = 8$ or $k = -4$

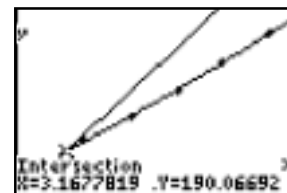
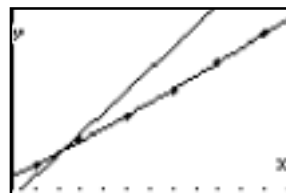
11. If $p^2 + 48 = -14p$, then $p^2 + 14p + 48 = 0$
 For this to be possible, the equation must have roots, (which correspond to the zeros of $f(p) = p^2 + 14p + 48$). The L.S. factors: $p^2 + 14p + 48 = (p + 6)(p + 8)$. So the equation becomes $(p + 6)(p + 8) = 0$, which clearly has two roots, $p = -6$ or $p = -8$. It is therefore possible for $p^2 + 48$ to equal $-14p$. This occurs when $p = -6$ and when $p = -8$.

12. No. A quadratic function has a graph which is a parabola. It is U-shaped and either opens up, down, right, or left. The graph only turns around once. This is why it will only cross the x -axis in at most two places.

13. When the data is entered in a graphing calculator and a quadratic regression is done, the equation produced is $y = 0.4x^2 + 31x + 87$.



According to the model, $C(x) = 0.4x^2 + 31x + 87$.
 If revenue is \$60 per item, the revenue function $R(x) = 60x$.
 If the plant breaks even, Revenue = Cost,
 $60x = 0.4x^2 + 31x + 87$
 Using a graphing calculator, you can enter $y = 60x$ in Y_2 and see where this intersects the cost function graph.



Using CALC intersect, the plant breaks even when $x \approx 3.168$. This represents a production level of 3168 items.
 Alternatively, solve the equation $0.4x^2 + 31x + 87 = 0$ using the quadratic formula.

$$14. s(t) = s_0 + v_0t - \frac{1}{2}gt^2$$

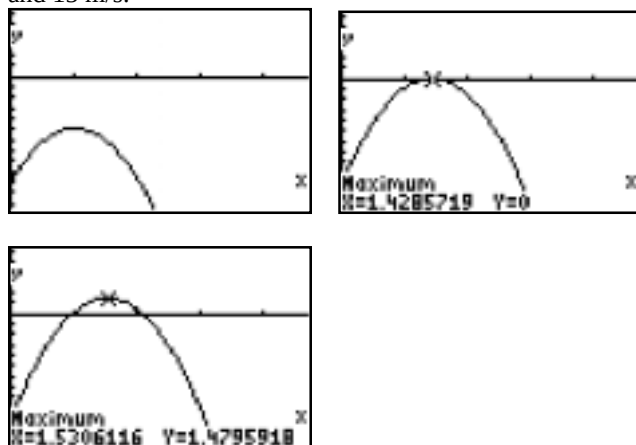
Since Tony is 10 metres below ground, $s_0 = -10$.
 $s(t) = -10 + v_0t - 4.9t^2$
 The graph of this quadratic function opens downward since $a < 0$. If the cell phone is to reach Eugenia, it must reach a height of at least one metre above ground, so she has a chance to catch it.
 If $s(t) = 1$, then $1 = -10 + v_0t - 4.9t^2$ and this equation must have at least one root. Rearrange the equation:
 $0 = -4.9t^2 + v_0t - 11$
 For this equation to have at least one root, $b^2 - 4ac \geq 0$.
 $v_0^2 - 4(-4.9)(-11) \geq 0$
 $v_0^2 - 215.6 \geq 0$
 $v_0^2 \geq 215.6$

Since Tony will throw the cell phone upwards, we assume $v_0 > 0$.
 So if $v_0^2 \geq 215.6$
 $v_0 \geq 14.68$

Tony must give the cell phone an initial velocity of at least 14.68 m/s if Eugenia is to have a chance to catch it.
 N.B. For the cell phone to just reach ground level, the equation $0 = -10 + v_0t - 4.9t^2$ must have at least one root.
 Then $v_0^2 - 4(-10)(-4.9) \geq 0$
 $v_0^2 - 196 \geq 0$
 $v_0^2 \geq 196$
 $v_0 \geq 14$

In this case, the cell phone has "reached" Eugenia but she is unlikely to catch it unless she stoops down and reaches into the excavation, which might be dangerous.
 An alternative approach is to use a graphing calculator and systematic trial with different values for v_0 .

The following graphs were produced with v_0 set to 10, 14, and 15 m/s.



If Tony throws the cell phone up at 10 m/s, it will not reach the top of the excavation.

With an initial velocity of 14 m/s, the cell phone just reaches the top of the excavation, but Eugenia only has a split second to catch it.

If Tony throws the cell phone with initial velocity 15 m/s, it reaches a height of almost 1.5 metres above the excavation and Eugenia has a full second to catch it.

15. (a) At the point(s) of intersection, $x^2 + y^2 = 25$ and $y = x - 1$. Substitute $y = x - 1$ into $x^2 + y^2 = 25$.

$$\begin{aligned} x^2 + (x - 1)^2 &= 25 \\ x^2 + x^2 - 2x + 1 &= 25 \text{ (Simplify)} \\ 2x^2 - 2x - 24 &= 0 \\ 2(x^2 - x - 12) &= 0 \text{ (Factor)} \\ 2(x - 4)(x + 3) &= 0 \\ x &= 4 \text{ or } x = -3 \text{ (Solve for } x) \end{aligned}$$

Determine corresponding y-values:

$$\begin{aligned} y &= 4 - 1 & y &= -3 - 1 \\ &= 3 & &= -4 \end{aligned}$$

Coordinates of the points of intersection are (4, 3) and (-3, 4).

(b) If the line $y = x + k$ is a tangent, then it only has one point in common with the circle $x^2 + y^2 = 25$. At this common point, $y = x + k$ and $x^2 + y^2 = 25$.

Substitute $y = x + k$ into $x^2 + y^2 = 25$

$$\begin{aligned} x^2 + (x + k)^2 &= 25 \\ x^2 + x^2 + 2kx + k^2 &= 25 \\ 2x^2 + 2kx + k^2 - 25 &= 0 \end{aligned}$$

If this equation only has one root, $b^2 - 4ac = 0$

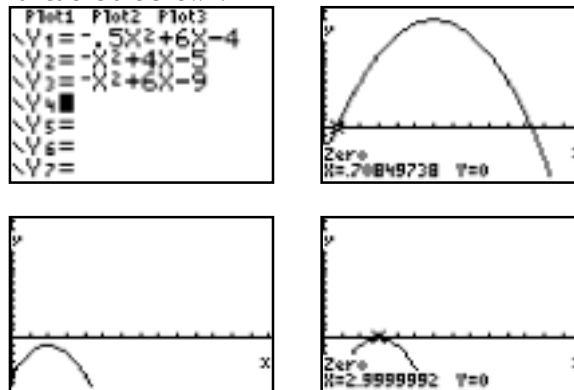
$$\begin{aligned} (2k)^2 - 4(2)(k^2 - 25) &= 0 \\ 4k^2 - 8k^2 + 200 &= 0 \\ 200 &= 4k^2 \\ 50 &= k^2 \\ k &= \pm 7.07 \end{aligned}$$

$y = x + k$ is a tangent to circle $x^2 + y^2 = 25$ when $k = \pm 7.07$.

16. The break-even points are the points at which costs are exactly balanced by revenue, so that neither a profit nor a loss is made. In other words, Revenue - Costs = 0.

(a) To find the break-even points graphically, create the profit function, $P(x) = R(x) - C(x)$, where $R(x)$ is the quadratic revenue function and $C(x)$ is the linear cost function. $P(x)$ will be a quadratic function. Graph the profit function and use the

CALC feature to find the break-even quantities, by finding the zeros of the profit function. The number of zeros will depend on the position of the graph. The graphs for 3 profit functions are shown.



In the first case, the company breaks even at $x = 0.7$ and is profitable for quite some time. There are two zeros.

The second case shows a non-profitable situation. The company does not break even. There are no zeros.

The third graph shows a case where the company just breaks even, when $x = 3$, but never makes a profit.

(b) To find the break-even points algebraically, find the zeros of the profit function by solving the quadratic equation that results from setting $P(x) = 0$.

For example, if the revenue function, $R(x) = -0.4x^2 + 10x$ and the cost function, $C(x) = 2x + 3$, the profit function, $P(x) = -0.4x^2 + 10x - 2x - 3 = -0.4x^2 + 8x - 3$.

The break-even quantities are the roots of the equation $-0.4x^2 + 8x - 3 = 0$

Using the quadratic formula,

$$\begin{aligned} x &= \frac{-8 \pm \sqrt{64 - 4(-0.4)(-3)}}{2(0.4)} \\ &= 0.382 \text{ or } 19.618 \end{aligned}$$

If x is measured in thousands, these quantities represent 382 items and 19 618 items. The number of zeros will depend on the sign of $b^2 - 4ac$. In this example, $b^2 - 4ac > 0$ and so there are two zeros. If $b^2 - 4ac = 0$, there is exactly one zero. If $b^2 - 4ac < 0$, there are no zeros.

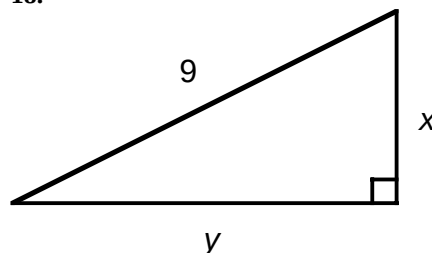
17. $f(x) = (k + 1)x^2 + 2kx + k - 1$

Use the discriminant $b^2 - 4ac$ to determine the number of zeros.

$$\begin{aligned} b^2 - 4ac &= (2k)^2 - 4(k + 1)(k - 1) \\ &= 4k^2 - 4(k^2 - 1) \\ &= 4k^2 - 4k^2 + 4 \\ &= 4 \end{aligned}$$

So $b^2 - 4ac > 0$ no matter what value k takes. The function has two zeros whatever the value of k .

18.



Let the other sides of the triangle have length x cm and y cm, as shown. If the area is 15 cm^2 , then

$$\frac{xy}{2} = 15$$

$$xy = 30$$

Also, by the Pythagorean Theorem, $x^2 + y^2 = 9^2$
 $x^2 + y^2 = 81$

$$\begin{aligned}\text{Now } (x + y)^2 &= x^2 + 2xy + y^2 \\ &= x^2 + y^2 + 2xy \\ &= 81 + 60 \\ &= 141\end{aligned}$$

$$\begin{aligned}\text{and } (x - y)^2 &= x^2 - 2xy + y^2 \\ &= x^2 + y^2 - 2xy \\ &= 81 - 60 \\ &= 21\end{aligned}$$

$$x + y = \sqrt{141} \text{ -----(1)}$$

$$x - y = \pm\sqrt{21} \text{ -----(2)}$$

We know x and y are positive since they represent lengths.

Therefore, $x + y$ is also positive, but $x - y$ might be positive or negative.

Add the two equations:

$$(1) + (2): \quad 2x = \sqrt{141} \pm \sqrt{21}$$

$$x \doteq 8.228 \text{ or } 3.646 \text{ (to 3 decimal places)}$$

Corresponding values of y may be obtained by subtracting the equations (1) – (2) or by using equation (1).

$$(1) - (2): \quad 2y = \sqrt{141} \pm \sqrt{21}$$

$$y \doteq 3.646 \text{ or } 8.228 \text{ (to 3 decimal places)}$$

$$\begin{aligned}\text{or (1): } \quad y &= \sqrt{141} - x \\ &= \sqrt{141} - 8.228 \\ &\doteq 3.646\end{aligned}$$

$$\begin{aligned}\text{or } \quad y &= \sqrt{141} - 3.646 \\ &\doteq 8.228\end{aligned}$$

The length of the other two sides are 3.65 cm and 8.23 cm (to 2 decimal places).

Practise, Apply, Solve 4.4, page 331

1. (a) Integer, rational, real, and complex
 (b) Rational, real, and complex
 (c) Complex
 (d) Irrational, real, and complex
 (e) Rational, real, and complex
 (f) Rational, real, and complex
 (g) Rational, real, and complex
 (h) Natural, whole, integer, rational, real, and complex

2. (a) $\sqrt{-16} = \sqrt{16i^2} = 4i$
 (b) $\sqrt{-121} = \sqrt{121i^2} = 11i$
 (c) $-\sqrt{-9} = -\sqrt{9i^2} = -3i$
 (d) $\sqrt{-18} = \sqrt{18i^2} = 3\sqrt{2}i$
 (e) $-\sqrt{-28} = -\sqrt{28i^2} = -2\sqrt{7}i$
 (f) $\pm\sqrt{-50} = \pm\sqrt{50i^2} = \pm5\sqrt{2}i$

3. In each case, simply change the sign of the “ i ” term.

- (a) $1 - i$
 (b) $2 + 3i$
 (c) $-3 + 4i$
 (d) $-7 - 5i$
 (e) $11 + 11i$
 (f) $2 - \sqrt{2}i$

4. Check each possible solution by substituting into the given equation and examining the values of the left side and right side.

$$(a) \quad x^2 = -16$$

<u>LS</u>	<u>RS</u>
$x^2 = (4i)^2$	-16
$= 16i^2$	
$= 16(-1)$	
$= -16$	

Yes, $4i$ is a solution.

$$(b) \quad x^2 = -14$$

<u>LS</u>	<u>RS</u>
$x^2 = (7i)^2$	-14
$= 49i^2$	
$= 49(-1)$	
$= -49$	

No, $7i$ is not a solution.

$$(c) \quad x^2 + 6 = 5$$

<u>LS</u>	<u>RS</u>
$x^2 + 6 = (i)^2 + 6$	5
$= -1 + 6$	
$= 5$	

Yes, i is a solution.

$$(d) \quad x^2 + 10 = 16$$

<u>LS</u>	<u>RS</u>
$x^2 + 10 = (3i)^2 + 10$	16
$= 9i^2 + 10$	
$= 9(-1) + 10$	
$= 1$	

No, $3i$ is not a solution.

5. (a)

$$x^2 + 4 = 0$$

$$\begin{aligned}x^2 &= -4 \\ &= 4(-1) \\ &= 4i^2 \\ x &= \pm\sqrt{4i^2} \\ &= \pm 2i\end{aligned}$$

(b)

$$x^2 + 64 = 0$$

$$\begin{aligned}x^2 &= -64 \\ &= 64(-1) \\ &= 64i^2 \\ x &= \pm\sqrt{64i^2} \\ &= \pm 8i\end{aligned}$$