

Quadratic equations that do not have real roots do have solutions in the set of complex numbers. The complex roots are always in conjugate pairs. If one root of a quadratic equation is $a + bi$, then the other root is $a - bi$. For example, consider the equation $2x^2 + 3x + 4 = 0$. The discriminant $b^2 - 4ac = 3^2 - 4(2)(4) = -23$.

Since this is less than zero, we know the equation has no roots in the Real number system. If we work in the Complex number system, we can use the fact that $i^2 = -1$ to solve the equation.

$$2x^2 + 3x + 4 = 0$$

$$x = \frac{-3 \pm \sqrt{-23}}{4}$$

$$= \frac{-3 \pm \sqrt{23i^2}}{4}$$

$$\doteq \frac{-3 \pm 4.796i}{4}$$

$$\doteq -0.75 \pm 1.20i \text{ (to 2 decimal places)}$$

The solutions to $2x^2 + 3x + 4 = 0$ are $-0.75 + 1.20i$ and $-0.75 - 1.20i$.

19. $x^4 - 81 = 0$ Factor the left side as much as possible

$$(x^2)^2 - 9^2 = 0 \text{ A difference of squares}$$

$$(x^2 - 9)(x^2 + 9) = 0$$

$$(x - 3)(x + 3)(x^2 + 9) = 0$$

$$\therefore x = 3, -3 \text{ or } x^2 + 9 = 0$$

$$x^2 = -9$$

$$x^2 = 9i^2$$

$$x = \pm 3i$$

Roots of the equation $x^4 - 81 = 0$ are $\pm 3, \pm 3i$.

20. No, you cannot put complex numbers in order. Although the real parts of a complex number could be put in increasing or decreasing order, the multiples of i cannot. To do this, we would need to set $i > 0$ or $i < 0$.

But if $i > 0$ and we multiply both sides by i we get $i^2 > 0$, which is not true ($i^2 < 0$). If $i < 0$ and we multiply both sides by i we get $i^2 > 0$ again, since multiplying by a negative number reverses the inequality sign.

21. Using the distributive property of multiplication over addition, $(x + 1)(x^2 - x + 1) = x^3 - x^2 + x + x^2 - x + 1$

$$= x^3 + 1$$

$$x^3 + 1 = 0$$

$$(x + 1)(x^2 - x + 1) = 0$$

$$x = -1 \text{ or } x^2 - x + 1 = 0$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3i^2}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

The roots of equation $x^3 + 1 = 0$ are $-1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$ (or $-1.00, 0.50 + 0.87i, 0.50 - 0.87i$).

Practise, Apply, Solve 4.5U, page 336

$$\begin{aligned} \text{1. (a)} \quad 2 + 6i + 5 - i &= (2 + 5) + (6 - 1)i \\ &= 7 + 5i \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 4 - 3i + 1 + i &= (4 + 1) + (-3 + 1)i \\ &= 5 - 2i \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad 3 + 7i + (-4) - 9i &= (3 - 4) + (7 - 9)i \\ &= -1 - 2i \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad -1 + 3i + 2 - 13i &= (-1 + 2) + (3 - 13)i \\ &= 1 - 10i \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad 5i + 1 - 2i &= (0 + 1) + (5 - 2)i \\ &= 1 + 3i \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad 3 + \sqrt{2}i + \sqrt{2}i &= (3 + 0) + (\sqrt{2} + \sqrt{2})i \\ &= 3 + 2\sqrt{2}i \end{aligned}$$

$$\begin{aligned} \text{2. (a)} \quad 2 + 6i - (5 - i) &= (2 - 5) + (6 + 1)i \\ &= -3 + 7i \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 4 - 3i - (1 + i) &= (4 - 1) + (-3 - 1)i \\ &= 3 - 4i \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad 3 + 7i - (-4 - 9i) &= (3 + 4) + (7 + 9)i \\ &= 7 + 16i \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad -1 + 3i - (2 - 13i) &= (-1 - 2) + (3 + 13)i \\ &= -3 + 16i \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad 5i - (1 - 2i) &= (0 - 1) + (5 + 2)i \\ &= -1 + 7i \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad 3 + \sqrt{2}i - \sqrt{2}i &= (3 + 0) + (\sqrt{2} - \sqrt{2})i \\ &= 3 + 0i \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{3. (a)} \quad (2 + i)(3 + 5i) &= 6 + 10i + 3i + 5i^2 \\ &= 6 + 13i - 5 \\ &= 1 + 13i \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (5 - 3i)(7 + i) &= 35 + 5i - 21i - 3i^2 \\ &= 35 - 16i + 3 \\ &= 38 - 16i \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad (4 - 3i)(-2 - i) &= -8 - 4i + 6i + 3i^2 \\ &= -8 + 2i - 3 \\ &= -11 + 2i \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad i(5 + i) &= 5i + i^2 = 5i - 1 \\ &= -1 + 5i \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad (2 + i)^2 &= 4 + 4i + i^2 \\ &= 4 + 4i - 1 \\ &= 3 + 4i \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad (2 + 3i)(2 - 3i) &= 4 - 9i^2 \\ &= 4 + 9 \\ &= 13 \text{ (or } 13 + 0i) \end{aligned}$$

4. (a)

$$\begin{aligned}z\bar{z} &= (1+i)(1-i) \\&= 1-i^2 \\&= 1+1 \\&= 2 \text{ (or } 2+0i\text{)}\end{aligned}$$

(b)

$$\begin{aligned}z\bar{z} &= (3-2i)(3+2i) \\&= 9-4i^2 \\&= 9+4 \\&= 13 \text{ (or } 13+0i\text{)}\end{aligned}$$

(c)

$$\begin{aligned}z\bar{z} &= (-4+2i)(-4-2i) \\&= 16-4i^2 \\&= 16+4 \\&= 20 \text{ (or } 20+0i\text{)}\end{aligned}$$

5. (a) $(3+4i) + (2+3i) = (3+2) + (4+3)i$
 $= 5+7i$

(b) $(5+i) - (3-2i) = (5-3) + (1-(-2))i$
 $= 2+3i$

(c) $(2i)^2 = 4i^2 = -4$ (or $-4+0i$)

(d) $i^4 = (i^2)^2 = (-1)^2 = 1$

(e) $\frac{1}{i^2} = \frac{1}{-1} = -1$

(f)

$$\begin{aligned}\frac{1}{(1+i)(1-i)} &= \frac{1}{1-i^2} \\&= \frac{1}{1+1} \\&= \frac{1}{2}\end{aligned}$$

(g) $(1+2i)^2 = (1+2i)(1+2i)$
 $= 1+4i+4i^2$
 $= 1+4i-4$
 $= -3+4i$

(h) $i(4-5i)^2 = i(4-5i)(4-5i)$
 $= i(16-40i+25i^2)$
 $= i(16-40i-25)$
 $= i(-9-40i)$
 $= -9i+40$
 $= 40-9i$

(i) $(2-i)(1+2i)(1-i)$
 $= (2-i)(1-i+2i-2i^2)$
 $= (2-i)(1+i+2)$
 $= (2-i)(3+i)$
 $= 6+2i-3i-i^2$
 $= 6-i+1$
 $= 7-i$

(j)

$$\begin{aligned}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) \\&= \frac{2}{4} + \frac{2}{4}i + \frac{2}{4}i + \frac{2}{4}i^2 \\&= \frac{1}{2} + i - \frac{1}{2} \\&= i \text{ (or } 0+i\text{)}\end{aligned}$$

(k) $(6-4i)^2 - (1-3i)^2 + 5i$
 $= 36-48i+16i^2 - (1-6i+9i^2) + 5i$
 $= 36-48i-16-1+6i+9+5i$
 $= 28-37i$

(l)

$$\begin{aligned}\sqrt{-4} + \sqrt{-25} &= \sqrt{4i^2} + \sqrt{25i^2} \\&= \pm 2i \pm 5i \\&= (2+5)i, (2-5)i, (-2+5)i, \text{ or } (-2-5)i \\&= 7i, -3i, 3i, \text{ or } -7i\end{aligned}$$

6. (a) $(x-2i)(x+2i) = 0$
 $x^2 - 4i^2 = 0$
 $x^2 + 4 = 0$

(b) $(x-(1+3i))(x-(1-3i)) = 0$
 $(x-1-3i)(x-1+3i) = 0$
 $(x-1)^2 - 9i^2 = 0$
 $x^2 - 2x + 1 + 9 = 0$
 $x^2 - 2x + 10 = 0$

(c) $(x-(3+4i))(x-(3-4i)) = 0$
 $(x-3-4i)(x-3+4i) = 0$
 $(x-3)^2 - 16i^2 = 0$
 $x^2 - 6x + 9 + 16 = 0$
 $x^2 - 6x + 25 = 0$

(d)

$$\begin{aligned}\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) &= 0 \\ \left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right) &= 0 \\ \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2 &= 0 \\ x^2 + x + \frac{1}{4} - \frac{3}{4}i^2 &= 0 \\ x^2 + x + 1 &= 0\end{aligned}$$

7. If one root is $3-i$, the other root is its complex conjugate $3+i$. Equation is:

$$\begin{aligned}(x-(3-i))(x-(3+i)) &= 0 \\ (x-3+i)(x-3-i) &= 0 \\ (x-3)^2 - i^2 &= 0 \\ x^2 - 6x + 9 + 1 &= 0 \\ x^2 - 6x + 10 &= 0\end{aligned}$$

$$8. x^2 - 4x + 5 = 0$$

Substitute $x = 2 + i$:

$$\begin{aligned} \text{LS} &= (2 + i)^2 - 4(2 + i) + 5 \\ &= 4 + 4i + i^2 - 8 - 4i + 5 \\ &= 4 + 4i - 1 - 8 - 4i + 5 \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

$$\text{RS} = 0$$

$\therefore x = 2 + i$ is a solution.

$$9. f(x) = 2x^2 + 3x - 2$$

$$\begin{aligned} f(2 - i) &= 2(2 - i)^2 + 3(2 - i) - 2 \\ &= 2(4 - 4i + i^2) + 6 - 3i - 2 \\ &= 2(4 - 4i - 1) + 4 - 3i \\ &= 2(3 - 4i) + 4 - 3i \\ &= 6 - 8i + 4 - 3i \\ &= 10 - 11i \end{aligned}$$

$$10. g(x) = -3x^2 - 2x + 4$$

$$\begin{aligned} g(3 - 4i) &= -3(3 - 4i)^2 - 2(3 - 4i) + 4 \\ &= -3(9 - 24i + 16i^2) - 6 + 8i + 4 \\ &= -3(9 - 24i - 16) - 6 + 8i + 4 \\ &= -3(-7 - 24i) - 2 + 8i \\ &= 21 + 72i - 2 + 8i \\ &= 19 + 80i \end{aligned}$$

11. A quadratic equation of the form $ax^2 + bx + c = 0$ where $b^2 - 4ac < 0$ will have complex solutions of the form $a + bi$. $x^2 + 4x + 29 = 0$ is such an equation.

Solutions: (using the quadratic formula)

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(29)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{-100}}{2} \\ &= \frac{-4 \pm 10i}{2} \\ &= -2 \pm 5i \end{aligned}$$

Solutions are $x = -2 + 5i$ and $x = -2 - 5i$.

To verify these:

When $x = -2 + 5i$

$$\begin{aligned} \text{LS} &= (-2 + 5i)^2 + 4(-2 + 5i) + 29 \\ &= 4 - 20i + 25i^2 - 8 + 20i + 29 \\ &= 4 - 20i - 25 - 8 + 20i + 29 \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

$$\text{RS} = 0$$

$\therefore x = -2 + 5i$ is a solution.

When $x = -2 - 5i$

$$\begin{aligned} \text{LS} &= (-2 - 5i)^2 + 4(-2 - 5i) + 29 \\ &= 4 + 20i + 25i^2 - 8 - 20i + 29 \\ &= 4 + 20i - 25 - 8 - 20i + 29 \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

$$\text{RS} = 0$$

$\therefore x = -2 - 5i$ is a solution.

12. If $a + bi = 0$, $a, b \neq 0$, then $a = -bi$

$$-\frac{a}{b} = i$$

But a, b are real numbers and $a, b \neq 0$, so $-\frac{a}{b}$ is also a real number (which is not zero). We know that i is not a real number so this is a contradiction. $a + bi$ can only be zero if both a and b are zero.

13. If $z = a + bi$, then $\bar{z} = a - bi$

$$\begin{aligned} \therefore z + \bar{z} &= 10 & \text{and} & & z - \bar{z} &= 14i \\ a + bi + a - bi &= 10 & & & a + bi - (a - bi) &= 14i \\ 2a &= 10 & & & 2bi &= 14i \\ a &= 5 & & & b &= 7 \end{aligned}$$

14. $z = -2 + 3i$ and $w = 4 - i$

$$\begin{aligned} z + w &= -2 + 3i + 4 - i \\ &= 2 + 2i \end{aligned}$$

$$\begin{aligned} z - w &= -2 + 3i - (4 - i) \\ &= -6 + 4i \end{aligned}$$

$$\begin{aligned} zw &= (-2 + 3i)(4 - i) \\ &= -8 + 2i + 12i - 3i^2 \\ &= -8 + 14i + 3 \\ &= -5 + 14i \end{aligned}$$

$$\begin{aligned} w^2 &= (4 - i)^2 \\ &= (4 - i)(4 - i) \\ &= 16 - 8i + i^2 \\ &= 16 - 8i - 1 \\ &= 15 - 8i \end{aligned}$$

$$\begin{aligned} z\bar{z} &= (-2 + 3i)(-2 - 3i) \\ &= 4 - 9i^2 \\ &= 4 + 9 \\ &= 13 \text{ or } (13 + 0i) \end{aligned}$$

15. Let $z = a + bi$ be a square root of $5 + 12i$.

$$\text{Then } (a + bi)^2 = 5 + 12i$$

$$a^2 + 2abi + b^2i^2 = 5 + 12i$$

$$a^2 - b^2 + 2abi = 5 + 12i$$

$$\text{So } a^2 - b^2 = 5 \quad \text{equating real parts}$$

$$\text{and } 2ab = 12 \quad \text{equating imaginary parts}$$

$$4a^2b^2 = 144$$

$$4a^2(a^2 - 5) = 144$$

$$4a^4 - 20a^2 - 144 = 0$$

$$4(a^4 - 5a^2 - 36) = 0$$

$$4(a^2 - 9)(a^2 + 4) = 0$$

$$4(a - 3)(a + 3)(a^2 + 4) = 0$$

$a = 3, -3$, or $a^2 + 4 = 0$ (we reject this since a is a real number)

$$2ab = 12$$

$$ab = 6$$

$$\text{If } a = 3, b = 2$$

$$\text{If } a = -3, b = -2$$

The square roots of $5 + 12i$ are $3 + 2i$ and $-3 - 2i$.

16. If $a + bi$ is a root of $x^2 + cx + d = 0$, then $a - bi$ is the other root. $\therefore$ another way to write the equation is

$$(x - (a + bi))(x - (a - bi)) = 0$$

$$(x - a - bi)(x - a + bi) = 0$$

$$(x - a)^2 - b^2 i^2 = 0$$

$$x^2 - 2ax + a^2 + b^2 = 0$$

But the equation is $x^2 + cx + d = 0$

$$\therefore -2a = c \text{ and } a^2 + b^2 = d$$

$$0 = 2a + c \quad \text{and} \quad a^2 + b^2 = d$$

Practise, Apply, Solve 4.6, page 345

1. (a) In table 1, $xy = 48$ for each row of the table. So

$$y = \frac{48}{x} \text{ and } x = \frac{48}{y}. \text{ This is an example of inverse variation.}$$

(b) In table 2, $xy = 3600$ for each row of the table. So

$$y = \frac{3600}{x} \text{ and } x = \frac{3600}{y}. \text{ This is an example of inverse}$$

variation.

(c) In table 3, y increases as x increases so this is not an inverse variation. In fact, $y = x^2$ describes the relationship between x and y . This is a quadratic relationship.

2. (a) $xy = 48$, or $y = \frac{48}{x}$

(b) $xy = 3600$, or $y = \frac{3600}{x}$

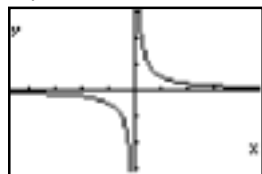
(c) $y = x^2$

3. (a) $f(x) = 2x$

i. $y = \frac{1}{f(x)} = \frac{1}{2x}$

ii. Vertical asymptotes occur where $2x = 0$. $\therefore$ equation of vertical asymptote is $x = 0$.

iii.



(b) $f(x) = x + 5$

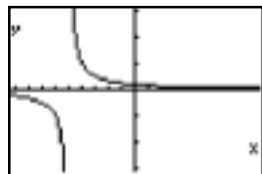
i. $y = \frac{1}{f(x)} = \frac{1}{x + 5}$

ii. $x + 5 = 0$

$$x = -5$$

Equation of vertical asymptote is $x = -5$.

iii.



(c) $f(x) = x - 4$

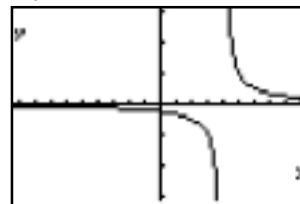
i. $y = \frac{1}{f(x)} = \frac{1}{x - 4}$

ii. $x - 4 = 0$

$$x = 4$$

Equation of vertical asymptote is $x = 4$.

iii.



(d) $f(x) = 2x + 5$

i. $y = \frac{1}{f(x)} = \frac{1}{2x + 5}$

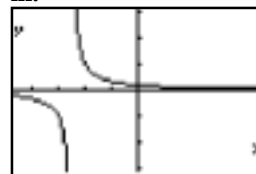
ii. $2x + 5 = 0$

$$2x = -5$$

$$x = -2.5$$

Equation of vertical asymptote is $x = -2.5$.

iii.



(e) $f(x) = -3x + 6$

i. $y = \frac{1}{f(x)} = \frac{1}{-3x + 6}$

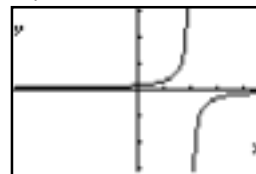
ii. $-3x + 6 = 0$

$$-3x = -6$$

$$x = 2$$

Equation of vertical asymptote is $x = 2$.

iii.



(f) $f(x) = (x - 3)^2$

i.

$$y = \frac{1}{f(x)}$$

$$y = \frac{1}{(x - 3)^2}$$

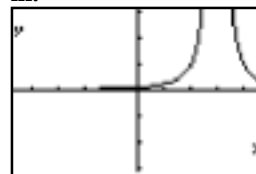
ii. $(x - 3)^2 = 0$

$$x - 3 = 0$$

$$x = 3$$

Equation of vertical asymptote is $x = 3$.

iii.



(g) $f(x) = x^2 - 3x - 10$

i. $y = \frac{1}{f(x)} = \frac{1}{x^2 - 3x - 10}$